Answers to Even-Numbered Exercises

CHAPTER 1

Section 1.1, page 10

2. $(3, -2)$ **4.** $(12, 3/2)$ **6.** $(12/7, 1/7)$

8. Either add 2 times row 3 to row 2, or add -3 times row 3 to row 1.

10. The solution set is empty. **12.** $(-24, 6, 5, -1)$

14. No solution **16.** $(-3, -1, 1)$

Note: The matrices you obtain in Exercises 17–20 may differ from the ones listed here. But your conclusions about the consistency of the systems should be the same as the answers given here.

18. Row equivalent to $\begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$, hence inconsistent.

20. Row equivalent to $\begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -3/2 & 3 \\ 0 & 0 & 0 & -5 & 15 \end{bmatrix}$, hence consistent.

22. All h **24.** $h = 6$ **26.** $-5g + 4h + k = 0$

28. Yes. The augmented matrix reduces to $\begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$, which corresponds to a consistent system of equations.

30. Interchange rows 1 and 2; interchange rows 1 and 2.

32. Add -3 times row 1 to row 3; add 3 times row 1 to row 3.

Answers for both odd- and even-numbered True–False questions will be included throughout this Instructor's Edition. (For odd-numbered exercises, the student *Study Guide* tells only where to look for answers.)

33. **a.** True. See the remarks following the box titled *Elementary Row Operations*.
- **b.** False. A 5×6 matrix has five rows.
- **c.** False. The description given applies to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. Mark a statement True only if the statement is *always* true.
- **d.** True. See the box before Example 2.

34. **a.** True. See the box preceding the subsection titled *Existence and Uniqueness Questions*.
- **b.** False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.
- **c.** False. By definition, an inconsistent system has *no* solution.
- **d.** True. This definition of *equivalent systems* is in the second paragraph after equation (2).

36. **[M]** $T_1 = 17.1°$, $T_2 = 21.4°$, $T_3 = 27.1°$, $T_4 = 17.1°$, $T_5 = 21.4°$, and $T_6 = 27.1°$. In rational form: $T_1 = 120/7$, $T_2 = 150/7$, $T_3 = 190/7$, $T_4 = 120/7$, $T_5 = 150/7$, and $T_6 = 190/7$.

Section 1.2, page 25

2. Reduced echelon: a, d; only echelon: b

4. Reduced echelon: none; only echelon: b

6. a. $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, pivot cols 1, 2

b. $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, pivot cols 1, 2, 3, 4

8. $\begin{cases} x_1 = 4 \\ x_2 = 3 \\ x_3 \text{ is free} \end{cases}$

10. $\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = -4 \end{cases}$

12. $\begin{cases} x_1 = -2 \\ x_2 = 3 \end{cases}$

14. Inconsistent

16. $\begin{cases} x_1 = 3 + 5x_3 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

18. All h **20.** All h

22. a. $h = -6$, and $k \neq 2$
b. $h \neq -6$
c. $h = -6$ and $k = 2$

23. a. False. See Theorem 1.
b. False. See the second paragraph of the section.
c. True. Basic variables are defined after equation (4).
d. True. This statement is at the beginning of *Parametric Descriptions of Solution Sets.*
e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

24. a. False. See the statement preceding Theorem 1. Only the *reduced* echelon form is unique.
b. False. See the beginning of the subsection *Pivot Positions.* The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.
c. True. See the paragraph after Example 3.
d. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.
e. True. See the paragraph just before Example 4.

26. The system is inconsistent, because the pivot in column 5 means that there is a row of the form $[0 \quad 0 \quad 0 \quad 0 \quad 1]$. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has no solution.

28. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \quad \text{and so} \quad \begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \end{aligned}$$

No matter what values a, b, and c have, the solution exists and is unique.

30. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

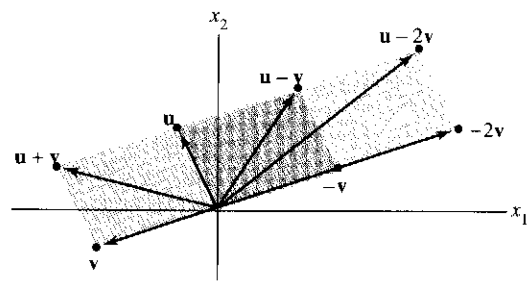
32. $\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$ **34.** $n = 30$: 5%; $n = 300$: .5%

36. **[M]** $p(t) = 1.7125t - 1.1948t^2 + .6615t^3 - .0701t^4 + .0026t^5$ (with coefficients rounded to four decimal places), and $p(7.5) = 64.6$ hundred lb. [*Note:* $p(7.5) = 64.8$ when the coefficients of $p(t)$ are retained as originally computed.] If a polynomial of lower degree were used for the interpolation, then the system of equations would be overdetermined. For the data given, a solution would not exist.

Section 1.3, page 36

2. $\begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

4.



6. $\begin{aligned} 2x_1 - x_2 - 7x_3 &= 0 \\ 3x_1 + 4x_2 + x_3 &= 0 \end{aligned}$

8. $\mathbf{w} = -2\mathbf{u} + \mathbf{v}$, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$, $\mathbf{y} = -3.5\mathbf{u} + 2\mathbf{v}$, $\mathbf{z} = -4\mathbf{u} + 3\mathbf{v}$. Yes, every vector in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

10. $x_1\begin{bmatrix}1\\5\\3\end{bmatrix} + x_2\begin{bmatrix}6\\0\\-1\end{bmatrix} + x_3\begin{bmatrix}2\\-6\\4\end{bmatrix} = \begin{bmatrix}5\\7\\1\end{bmatrix}$

12. No **14.** Yes

16. Noninteger weights are acceptable, of course, but some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix},\quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$2\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix}-4\\0\\2\end{bmatrix},\quad 0\cdot\mathbf{v}_1 + 2\cdot\mathbf{v}_2 = \begin{bmatrix}2\\0\\4\end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}-1\\0\\3\end{bmatrix},\quad 2\cdot\mathbf{v}_1 + 2\cdot\mathbf{v}_2 = \begin{bmatrix}-2\\0\\6\end{bmatrix}$$

18. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix}-3.7\\-.4\\11.2\end{bmatrix},\quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}5.8\\2.1\\5.3\end{bmatrix}$$

$$2\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix}-7.4\\-.8\\22.4\end{bmatrix},\quad 0\cdot\mathbf{v}_1 + 2\cdot\mathbf{v}_2 = \begin{bmatrix}11.6\\4.2\\10.6\end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix}2.1\\1.7\\16.5\end{bmatrix},\quad 2\cdot\mathbf{v}_1 + 2\cdot\mathbf{v}_2 = \begin{bmatrix}4.2\\3.4\\33.0\end{bmatrix}$$

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the xz-plane in ordinary 3-space, because Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ consists of all vectors of the form $(x, 0, z)$.

22. $h = -2$

24. Let $\mathbf{w} = \begin{bmatrix}h\\k\end{bmatrix}$. Then $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$ if and only if the vector equation $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$ is consistent. Row reduce the augmented matrix:

$$\begin{bmatrix}2 & 1 & h\\-6 & 5 & k\end{bmatrix} \sim \begin{bmatrix}2 & 1 & h\\0 & 8 & k+3h\end{bmatrix}$$

This system is consistent for all h and k. (See Theorem 2 in Section 1.2.)

26. a. Yes **b.** $\mathbf{a}_3 = 0\cdot\mathbf{a}_1 + 0\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3$

28. a. $27.6x_1 + 30.2x_2$ million Btu

b. $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix} + x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}$

c. [M] $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix} + x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix} = \begin{bmatrix}162\\23{,}610\\1{,}623\end{bmatrix}$

3.9 tons of anthracite coal and 1.8 tons of bituminous coal

30. Let $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{w} = (w_1, \dots, w_n)$.

a. For $j = 1, \dots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in $\mathbb{R}$, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ equals $\mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For $j = 1, \dots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of $\mathbb{R}$. By vector equality, $\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}$.

c. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in $\mathbb{R}$. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

d. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in $\mathbb{R}$ are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

31. a. False. The alternative notation is $(-4, 3)$, using parentheses and commas.

b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix}-5\\2\end{bmatrix}$ were on the line through $\begin{bmatrix}-2\\5\end{bmatrix}$ and the origin, then $\begin{bmatrix}-5\\2\end{bmatrix}$ would have to be a multiple of $\begin{bmatrix}-2\\5\end{bmatrix}$, which is not the case.

c. True. See the line displayed just before Example 4.

d. True. See the box that discusses the matrix in (5).

e. False. Span$\{\mathbf{u}, \mathbf{v}\}$ is often a plane through the origin, but Span$\{\mathbf{u}, \mathbf{v}\}$ is not a plane when $\mathbf{v}$ is a multiple of $\mathbf{u}$, or when $\mathbf{u}$ is the zero vector.

32. a. True. See the beginning of the subsection *Vectors in* $\mathbb{R}^n$.

b. True. Use Fig. 7 to draw the parallelogram determined by $\mathbf{u} - \mathbf{v}$ and $\mathbf{v}$.

c. False. See the first paragraph of the subsection *Linear Combinations*.

d. True. See the statement that refers to Fig. 11.

e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Section 1.4, page 46

2. $A\mathbf{x} = 5\begin{bmatrix}4\\2\\0\end{bmatrix} + 2\begin{bmatrix}1\\-5\\6\end{bmatrix} = \begin{bmatrix}20\\10\\0\end{bmatrix} + \begin{bmatrix}2\\-10\\12\end{bmatrix} = \begin{bmatrix}22\\0\\12\end{bmatrix}$,

and $A\mathbf{x} = \begin{bmatrix}4\cdot5 + 1\cdot2\\2\cdot5 - 5\cdot2\\0\cdot5 + 6\cdot2\end{bmatrix} = \begin{bmatrix}22\\0\\12\end{bmatrix}$

4. The product is not defined.

6. $A\mathbf{x} = r\begin{bmatrix}0\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}t\\s\\r\end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix}0\cdot r + 0\cdot s + 1\cdot t\\0\cdot r + 1\cdot s + 0\cdot t\\1\cdot r + 0\cdot s + 0\cdot t\end{bmatrix} = \begin{bmatrix}t\\s\\r\end{bmatrix}$$

8. $\begin{bmatrix}0 & 3 & -2 & 1\\1 & -2 & 6 & 0\\7 & 1 & -5 & -8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

10. $x_1\begin{bmatrix}0\\1\\-3\end{bmatrix} + x_2\begin{bmatrix}5\\2\\-1\end{bmatrix} + x_3\begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

12. $\mathbf{x} = \begin{bmatrix}3/5\\-4/5\\1\end{bmatrix}$

14. $-3\begin{bmatrix}6\\1\\-5\\9\end{bmatrix} - 7\begin{bmatrix}-2\\3\\4\\-3\end{bmatrix} = \begin{bmatrix}-4\\-24\\-13\\-6\end{bmatrix}$

16. $\begin{bmatrix}3 & -5 & 0\\1 & 2 & -1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}-4\\7\end{bmatrix}$

18. No, $\mathbf{u}$ is not in the plane spanned by the columns of A because the equation $A\mathbf{x} = \mathbf{u}$ has no solution. Row reduction of the augmented matrix $[\,A \quad \mathbf{u}\,]$ reveals an inconsistent equation:

$$\begin{bmatrix}4 & 3 & 5 & 8\\0 & 1 & -1 & 2\\1 & 2 & 0 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0 & 3\\0 & 1 & -1 & 2\\0 & -5 & 5 & -4\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0 & 3\\0 & 1 & -1 & 2\\0 & 0 & 0 & 6\end{bmatrix}$$

20. The equation $A\mathbf{x} = \mathbf{b}$ is not consistent unless $-b_1 + 2b_2 + b_3 = 0$. The set of $\mathbf{b}$ such that the equation *is* consistent is a plane through the origin: $-x_1 + 2x_2 + x_3 = 0$.

22. 4 rows. There is one pivot in each of the four pivot columns, and each pivot is in a different row.

24. The columns of A do not span $\mathbb{R}^4$, by Theorem 4, because an echelon form of A has a row of zeros, with pivot positions in only 3 rows.

26. The columns of the 3×2 matrix from Exercise 17 do not span $\mathbb{R}^3$ because with only two columns in the matrix, there can be at most two pivot positions, not enough for a pivot in each of the three rows. By Theorem 4, the columns do not span $\mathbb{R}^3$.

28. Yes, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$, by Theorem 4, because the matrix $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]$ has a pivot in each row, as the following row reduction shows:

$$\begin{bmatrix}1 & -1 & 3\\0 & 3 & -2\\-1 & 7 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 3\\0 & 3 & -2\\0 & 6 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 3\\0 & 3 & -2\\0 & 0 & 5\end{bmatrix}$$

29. **a.** False. See the paragraph following equation (3). The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *matrix equation*.
b. True. See the box before Example 2.
c. False. See the warning following Theorem 4.
d. True. See Example 4.
e. True. See parts (b) and (a) in Theorem 4.
f. True. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

30. **a.** False. See the paragraph following equation (3). The equation $A\mathbf{x} = \mathbf{b}$ involves vectors, but it is called a *matrix equation*.
b. True. See Example 2.
c. True, by Theorem 4.
d. True. See the box before Example 2. Saying that $\mathbf{b}$ is not in the set spanned by the columns of A is the same as saying that $\mathbf{b}$ is not a linear combination of the columns of A.
e. False. See the warning that follows Theorem 4.
f. True. In Theorem 4, statement (b) is false if and only if statement (a) is also false.

32. $2\begin{bmatrix}3\\8\\4\end{bmatrix} - 5\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix}$, so $x_1 = 2$, $x_2 = -5$.

34. $5\mathbf{u} = \begin{bmatrix}5\\15\\15\end{bmatrix}$, $A(5\mathbf{u}) = \begin{bmatrix}5 & 1 & -3\\7 & -2 & 1\end{bmatrix}\begin{bmatrix}5\\15\\15\end{bmatrix} = \begin{bmatrix}-5\\20\end{bmatrix}$

$$A\mathbf{u} = \begin{bmatrix}5 & 1 & -3\\7 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\3\\3\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix},\ 5(A\mathbf{u}) = \begin{bmatrix}-5\\20\end{bmatrix}$$

36. Use the fact that $A(4\mathbf{y}) = 4A\mathbf{y}$. **38.** No; no.

40. **[M]** The columns do not span $\mathbb{R}^4$.

42. **[M]** The columns span $\mathbb{R}^4$.

44. **[M]** Delete column 3 of the matrix in Exercise 42; no.

Section 1.5, page 55

2. Yes, there are at most three basic variables. **4.** No

. $x_3 \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

$x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

The set is a "line" in $\mathbb{R}^4$ through $\begin{bmatrix} 7 \\ -5 \\ 1 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} -8 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$. The solution set is a line through $\begin{bmatrix} -8 \\ 4 \\ 0 \end{bmatrix}$, parallel to the solution set of the homogeneous quations in Exercise 6.

ct $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$. The solution set f the homogeneous equation is Span$\{\mathbf{u}, \mathbf{v}\}$, the plane panned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous quation is the translated plane $\mathbf{p}$ + Span$\{\mathbf{u}, \mathbf{v}\}$, which asses through $\mathbf{p}$.

$= \mathbf{a} + t\mathbf{b}$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$,

$\begin{cases} x_1 = -4 + 2t \\ x_2 = -3t \end{cases}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix} + t \begin{bmatrix} -7 \\ 4 \end{bmatrix}$, or $\begin{cases} x_1 = 9 - 7t \\ x_2 = -5 + 4t \end{cases}$

True. See the first paragraph of the subsection titled *Homogeneous Linear Systems*.

False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the first two sentences of the subsection titled *Parametric Vector Form*.

False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

False. The line goes through $\mathbf{p}$ parallel to $\mathbf{v}$. See the paragraph that precedes Fig. 5.

e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector $\mathbf{p}$ such that $A\mathbf{p} = \mathbf{b}$.

22. a. False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero $\mathbf{x}$ that satisfies the equation. See the sentence before Example 2.

b. True. See Example 2 and the paragraph following it.

c. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

d. True. See the paragraph following Example 3.

e. False. The statement is true only when the solution set of $A\mathbf{x} = \mathbf{b}$ is nonempty. Theorem 6 applies only to a consistent system.

24. $A\mathbf{w} = A(5\mathbf{u}) = 5A\mathbf{u} = 5 \cdot \mathbf{0} = \mathbf{0}$, by Theorem 5.

26. (*Geometric argument using Theorem 6.*) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

28. No; the "origin" is the vector $\mathbf{0}$, and $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$.

30. a. Yes **b.** No **32. a.** Yes **b.** Yes

34. One answer: $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

36. Take some other value for p_S, say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices. Changing the unit of measurement to, say, German marks has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

38. a.

Distribution of Output: Agriculture	Mining	Manufacturing	Purchased by:
.65	.20	.20	Agriculture
.05	.10	.30	Mining
.30	.70	.50	Manufacturing

b. **[M]** $x_1 = .7869x_3$, $x_2 = .3770x_3$, x_3 is free. The data probably justifies only two significant figures, so we can take $x_3 = 100$ and round off the other prices to get $x_1 = 79$ and $x_2 = 38$.

40. [M] $x_1\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\10\\35\\0\end{bmatrix}+x_3\begin{bmatrix}0\\1\\0\\0\\4\\2\end{bmatrix}$

$-x_4\begin{bmatrix}1\\0\\0\\0\\4\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\0\\0\\3\end{bmatrix}+x_6\begin{bmatrix}0\\3\\0\\1\\12\\0\end{bmatrix}+x_7\begin{bmatrix}0\\0\\0\\0\\1\\2\end{bmatrix}$

$(x_1,\ldots,x_7)=(16,13,374,16,26,130,327)$

Section 1.6, page 64

2. Let A be the matrix with the given vectors as columns, and row reduce the augmented matrix for the equation $A\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}-2&8&-1&0\\0&0&0&0\\0&-5&3&0\end{bmatrix}\sim\begin{bmatrix}-2&8&-1&0\\0&-5&3&0\\0&0&0&0\end{bmatrix}$$

The equation $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution because x_3 is a free variable. So the columns of A are linearly dependent.

4. Lin. dep. **6.** Lin. indep. **8.** Lin. dep.

10. Lin. indep. **12.** Lin. dep. **14. a.** All h **b.** All h

16. $h=-3$ **18.** All h **20.** Lin. indep.

22. Lin. dep. **24.** Lin. dep.

26. A: any 2×2 matrix with two nonzero columns that are not multiples of each other; B: any 2×2 matrix with one column a multiple of the other.

27. a. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.

b. False. See the warning after Theorem 7.

c. True. See Fig. 3, after Theorem 8.

d. True. See the remark following Example 4.

28. a. True. See Fig. 1.

b. False. For instance, the set consisting of $\begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\begin{bmatrix}2\\-4\\6\end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.

c. True. See the remark following Example 4.

d. False. See Example 3(a).

30. $\mathbf{x}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$

32. True, by Theorem 9.

34. False.

Counterexample: $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$,

$\mathbf{v}_4=\begin{bmatrix}3\\3\\3\\3\end{bmatrix}$

36. True. If the equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$ had a nontrivial solution (at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3+0\cdot\mathbf{v}_4=\mathbf{0}$. But that can't happen because $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 35, if you know that the statement in Exercise 35 is true.

38. a. n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x}=\mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x}=\mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

40. An $m\times n$ matrix A with n pivot columns has a pivot in each column. So the equation $A\mathbf{x}=\mathbf{b}$ has no free variables. If there is a solution, it must be unique.

42. [M] $B=\begin{bmatrix}12&10&-3&10\\-7&-6&7&5\\9&9&-5&-1\\-4&-3&6&9\\8&7&-9&-8\end{bmatrix}$ (columns 1, 2, 4, and 6 of A)

Other choices are possible: Use column 3 instead of column 2, or use column 5 instead of column 4.

44. [M] Each column of A that is not a column of B is in the set spanned by the columns of B. *Reason*: The original matrix A has only four pivot columns. If one or more columns of A are removed, the resulting matrix will have at most four pivot columns. (Use exactly the same row operations on the new matrix that were used to reduce A to echelon form.) If $\mathbf{v}$ is a column of A that is not in B, then row reduction of the augmented matrix $[\,B\ \ \mathbf{v}\,]$ shows that only the first four columns of $[\,B\ \ \mathbf{v}\,]$ are pivot columns, which implies that the equation $B\mathbf{x}=\mathbf{v}$ has a solution.

Section 1.7, page 73

2. $\begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 10 \\ -2 \\ 8 \end{bmatrix}$ **4.** $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, yes

6. $\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$, no **8.** 4 rows and 3 columns

10. $x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}$ **12.** Yes

14. A projection onto the x_2-axis

16. A reflection in the x_2-axis

18. An expansion (scaling) in the x_1-direction (only) by a factor of 2

20. $\begin{bmatrix} 21 \\ -9 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 17 \\ -9 \\ -5 \end{bmatrix}$

22. $A = \begin{bmatrix} -7 & 3 \\ 4 & -8 \end{bmatrix}$

23. a. True. Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. See the paragraph before Example 1.

c. True. See the paragraph following the box that contains equation (4).

d. True. See the paragraph following equation (5).

24. a. True. See the paragraph following the definition of a linear transformation.

b. False. The question is an existence question. See the remark about Example 1(d), following the example.

c. True. See the discussion following the definition of a linear transformation.

26. The plane is the set of all vectors of the form

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

Using property (4) of a linear transformation gives

$$T(\mathbf{x}) = T(s\mathbf{u} + t\mathbf{v}) = sT(\mathbf{u}) + tT(\mathbf{v}) \quad (s, t \text{ in } \mathbb{R})$$

If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, then the set of images $T(\mathbf{x})$ forms the plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are multiples, and if one of them is not zero, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then the image of every point is just the origin.

28. If $\mathbf{x}$ is in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$, for some a and b between 0 and 1, inclusive. Using the linearity of T, we get $T(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$. Since $0 \le a, b \le 1$, this formula for $T(\mathbf{x})$ shows that $T(\mathbf{x})$ lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. (A degenerate parallelogram is possible.)

30. $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$, so T cannot be a linear transformation.

32. Take any vector (x_1, x_2) with $x_2 \neq 0$, and use a negative scalar. For instance, $T(0, 1) = (-2, 3)$, but $T(-1 \cdot (0, 1)) = T(0, -1) = (2, 3) \neq (-1) \cdot T(0, 1)$.

34. Suppose that $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are linearly independent and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then there exist weights c_1, c_2, not both zero, such that

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$$

Since T is linear, $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. If the vector $c_1\mathbf{u} + c_2\mathbf{v}$ were zero, then c_1 and c_2 would both be zero, because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. However, c_1 and c_2 are not both zero, so we conclude that $c_1\mathbf{u} + c_2\mathbf{v}$ is a nonzero solution of $T(\mathbf{x}) = \mathbf{0}$.

36. [M] The set of all multiples of $(-4, 5, -7, 3)$.

38. [M] Yes, one choice for $\mathbf{x}$ is $\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, 0\right)$.

Section 1.8, page 83

2. $\begin{bmatrix} 3 & -2 \\ 3 & 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 3 \\ 2 & -6 \\ 0 & 1 \\ 5 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **16.** $\begin{bmatrix} 5 & -6 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$ **18.** $\begin{bmatrix} -1 & 5 \\ 0 & 0 \\ 3 & 1 \\ 1 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 3 & -4 & 0 & 8 \end{bmatrix}$ **22.** $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

23. a. True. See Theorem 10.

b. True. See Example 3.

c. False. See the definition of "onto" on p. 81. *Any* function from $\mathbb{R}^n$ to $\mathbb{R}^m$ maps each vector onto another vector.

24. a. False. See the paragraph preceding Example 2.

b. True. See Theorem 8.

c. False. See the definition on p. 81. Any *function* from $\mathbb{R}^n$ to $\mathbb{R}^m$ maps a vector onto a single vector.

26. Yes, by Theorem 12(b), because the columns of the matrix are linearly independent (since neither is a multiple of the other).

28. No, by Theorem 12(a). The columns of A cannot span $\mathbb{R}^4$ because with only two columns, A cannot have a pivot in each row. (See Theorem 4 in Section 1.4.)

30. The transformation is not one-to-one because the standard matrix has linearly dependent columns. The transformation does map $\mathbb{R}^3$ onto $\mathbb{R}^2$ because the standard matrix has a pivot position in each row.

32. m. A has m pivot columns if and only if A has a pivot position in each row. By Theorem 4, this happens if and only if the columns of A span $\mathbb{R}^m$, and this in turn happens, by Theorem 12, if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

34. The transformation T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if for each $\mathbf{y}$ in $\mathbb{R}^m$ *there exists* an $\mathbf{x}$ in $\mathbb{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.

36. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^p$ and scalars c and d. Then

$$\begin{aligned} &T(S(c\mathbf{u} + d\mathbf{v})) \\ &\quad = T(c \cdot S(\mathbf{u}) + d \cdot S(\mathbf{v})) && \text{Since } S \text{ is linear} \\ &\quad = c \cdot T(S(\mathbf{u})) + d \cdot T(S(\mathbf{v})) && \text{Since } T \text{ is linear} \end{aligned}$$

For $c = d = 1$, this calculation shows that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ preserves sums; for $d = 0$, the calculation shows that the mapping preserves scalar products. So the mapping is linear.

38. **[M]** No. There is no pivot in the third column.

40. **[M]** No. There is no pivot in the fifth row.

Section 1.9, page 92

2. a. $B = \begin{bmatrix} 110 & 110 \\ 3 & 2 \\ 21 & 25 \\ 3 & .4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b. **[M]** The required nutrients cannot be provided exactly; however, a good approximation is given by .2443 serving of Oat Bran and .7556 serving of Crispix. *Note:* .25 serving of Oat Bran plus .75 serving of Crispix supplies 110 calories, 2.25 g protein, 24 g carbohydrate, and 1.05 g fat.

4. $x_1 \begin{bmatrix} 10 \\ 50 \\ 30 \end{bmatrix} + x_2 \begin{bmatrix} 20 \\ 40 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} 100 \\ 300 \\ 200 \end{bmatrix}$, where x_1, x_2, and x_3 are the number of units of foods 1, 2, and 3, respectively, to be used in the meal.
[M] The solution is $x_1 = 150/33 \approx 4.55$, $x_2 = 50/33 \approx 1.52$, and $x_3 = 40/33 \approx 1.21$.

6. $$\begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 7 & -2 & 0 \\ 0 & -2 & 9 & -2 \\ 0 & 0 & -2 & 11 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}.$$

[M]: $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.56 \\ 8.90 \\ 4.59 \\ 1.74 \end{bmatrix}$

8. $$\begin{bmatrix} 15 & -5 & 0 & -5 & -1 \\ -5 & 15 & -5 & 0 & -2 \\ 0 & -5 & 15 & -5 & -3 \\ -5 & 0 & -5 & 15 & -4 \\ -1 & -2 & -3 & -4 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 10 \\ -40 \\ 40 \\ -50 \\ 0 \end{bmatrix}.$$

[M]: $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} -3.49 \\ -5.14 \\ -2.10 \\ -6.42 \\ -4.57 \end{bmatrix}$

10. $\mathbf{x}_{k+1} = M\mathbf{x}_k$, for $k = 0, 1, 2, \ldots$, where $M = \begin{bmatrix} .94 & .04 \\ .06 & .96 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$. The population in 1992 (when $k = 2$) is $\mathbf{x}_2 = \begin{bmatrix} 746{,}800 \\ 553{,}200 \end{bmatrix}$.

12. **[M]** $x_0 = \begin{bmatrix} 304 \\ 48 \\ 98 \end{bmatrix}$, $\mathbf{x}_1 \approx \begin{bmatrix} 307 \\ 48 \\ 95 \end{bmatrix}$, $\mathbf{x}_2 \approx \begin{bmatrix} 310 \\ 48 \\ 92 \end{bmatrix}$

14. **[M]** The interior temperatures are:

Fig. 3(a): (11.4°, 14.3°, 11.4°, 11.4°, 14.3°, 11.4°)
Fig. 3(b): (5.7°, 7.1°, 15.7°, 5.7°, 7.1°, 15.7°)

At each interior point in the figure for Exercise 36 in Section 1.1, the temperature is the sum of the temperatures at the corresponding interior points here in Fig. 3(a) and 3(b). If the boundary temperatures in Fig. 3(a) are changed by a factor of 3, then the interior temperatures should also change by a factor of 3.

In fact, solutions of the steady-state temperature problem here satisfy a superposition principle. The system of equations that approximate the interior temperatures can be written in the form $A\mathbf{x} = \mathbf{b}$, where A is determined by the arrangement of the interior points on the plate and $\mathbf{b}$ is determined by the boundary temperatures. If $\mathbf{b}_1$ and $\mathbf{b}_2$ are two sets of boundary temperatures, and if $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$, then $\mathbf{x}_1 + \mathbf{x}_2$ satisfies $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$. Also, for any constant c, the interior temperatures $c\mathbf{x}_1$ satisfy $A\mathbf{x} = c\mathbf{b}_1$. So the solution of this internal temperature problem is a linear function of the boundary temperatures.

CHAPTER 2

Section 2.1, page 107

2. $\begin{bmatrix} 6 & 4 & 0 \\ 4 & 2 & 2 \end{bmatrix}$, not defined, $\begin{bmatrix} 19 & -8 & 1 \\ 4 & -16 & -4 \end{bmatrix}$, not defined

4. $\begin{bmatrix} 3 & -5 & 3 \\ 5 & 6 & -2 \\ -3 & 2 & -2 \end{bmatrix}$, $\begin{bmatrix} 2 & -5 & 3 \\ 5 & 5 & -2 \\ -3 & 2 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 5 \\ 15 & -5 \\ 1 & 3 \end{bmatrix}$ **8.** 2 **10.** $k = 3$

12. Example: $\begin{bmatrix} 12 & 3 \\ 4 & 1 \end{bmatrix}$

14. AD is obtained by multiplying the columns of A by 2, 3, and 4, respectively. DA is obtained by multiplying the rows of A by 2, 3, and 4, respectively. Any diagonal matrix of the form λI where λ is a scalar commutes with A.

15. a. False. See the definition of AB.
b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.
c. True. See Theorem 2(b), read right to left.
d. True. See Theorem 3(b), read right to left.
e. False. If the phrase "in the reverse order" were added to the statement, it would be true. See the box after Theorem 3.

16. a. False. AB is also a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.
b. True. See the box after Example 6.
c. False. The left-to-right order of B and C cannot be changed, in general.
d. False. See Theorem 3(d).
e. True. This general statement follows from Theorem 3(b).

8. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

0. The second column of AB is also all zeros.

2. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

. If $AD = I_n$, then $\mathbf{y}$ satisfies $(AD)\mathbf{y} = \mathbf{y}$, and hence $A(D\mathbf{y}) = \mathbf{y}$. The vector $\mathbf{x} = D\mathbf{y}$ is a solution of $A\mathbf{x} = \mathbf{y}$.

. $\mathbf{u}^T\mathbf{v} = (\mathbf{v}^T\mathbf{u})^T = \mathbf{v}^T\mathbf{u}$, because $\mathbf{v}^T\mathbf{u}$ is a 1×1 matrix, which we regard as a scalar; $(\mathbf{v}^T\mathbf{u})^T = \mathbf{u}\mathbf{v}^T$

. The (i, j)-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because

$$r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}(ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$$

30. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n is $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and zeros elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

32. $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

34. [M] The answer will depend on your choice of software or graphing calculator. In MATLAB, the command `rand(6,4)` creates a 6×4 matrix with random entries uniformly distributed between 0 and 1. The command

```
round(9*(2*rand(6,4) - 1))
```

creates a random 6×4 matrix with integer entries between -9 and 9. The MATLAB Toolbox, on the data disk available from the publisher, contains a command `randint` that will produce a random matrix with integer entries.

By default, the TI-85 calculator command `randM(6,4)` creates a random 6×4 matrix with integer entries between -9 and 9.

36. [M] The equality $(AB)^T = A^TB^T$ is very likely to be false for 4×4 matrices selected at random.

38. [M] The matrices approach the matrix
$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Section 2.2, page 117

2. $\begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 4 & 9/2 \\ -3 & -7/2 \end{bmatrix}$ **6.** $\begin{bmatrix} 15 \\ 7 \end{bmatrix}$

8. [M] The times for the operation will depend on the type of commands available. If `inv(A)` and `rref(M)` are used, the method using `rref` will probably be faster.

9. a. True, by definition of *invertible*.
b. False. See Theorem 6(b).
c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1\cdot 1 - 0\cdot 0 = 1 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.
d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.
e. True, by the box just before Example 6.

10. a. False. The phrase "in the reverse order" is missing. See Theorem 6(b).

b. True, by Theorem 6(a).
c. True, by Theorem 4.
d. True, by Theorem 7.
e. False. The last part of Theorem 7 is misstated here.

12. $AC = I \Rightarrow A^{-1}AC = A^{-1}I$ (suppressing parentheses because of associativity) $\Rightarrow IC = A^{-1}$ and $C = A^{-1}$.

14. $(B - C)A = 0 \Rightarrow (B - C)AA^{-1} = 0A^{-1} \Rightarrow$
$(B - C)I = 0 \Rightarrow B - C = 0 \Rightarrow B = C$.

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1} \Rightarrow CB^{-1} = AI \Rightarrow A = CB^{-1}$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6. [*Note:* You cannot use the formula $(AB)^{-1} = B^{-1}A^{-1}$ in the proof because you cannot assume that A is invertible.]

18. $B = P^{-1}AP$

20. Take transposes in the equations $AA^{-1} = I$ and $A^{-1}A = I$ to obtain

$$(A^{-1})^T A^T = I^T = I \quad \text{and} \quad A^T(A^{-1})^T = I$$

These equations say that A^T is invertible, with inverse $(A^{-1})^T$.

22. If an $n \times n$ matrix A is invertible, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$, by Theorem 5. Hence the columns of A span $\mathbb{R}^n$, by Theorem 4 in Section 1.4.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the n pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

26. $$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

28. a. If E is formed by interchanging rows r and s of I_4, then left-multiplication by E interchanges rows r and s of A. The inverse F of E is E itself.

b. $$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and }$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-multiplication by E multiplies row 3 of A by 5.

30. $$\begin{bmatrix} -1 & 2 \\ -1/3 & 1 \end{bmatrix}$$

32. $$\begin{bmatrix} -28 & -13 & 3 \\ 2 & 1 & 0 \\ -7 & -3 & 1 \end{bmatrix}$$

34. Not invertible

36. Row reduce $\begin{bmatrix} -25 & -9 & -27 & 0 & 0 \\ 546 & 180 & 537 & 1 & 0 \\ 154 & 50 & 149 & 0 & 1 \end{bmatrix}$ and obtain the last two columns of:

$$A^{-1} = \begin{bmatrix} 5 & 3/2 & -9/2 \\ -224 & -433/6 & 439/2 \\ 70 & 68/3 & -69 \end{bmatrix} = \begin{bmatrix} 5 & 1.5000 & -4.5000 \\ -224 & -72.1667 & 219.5000 \\ 70 & 22.6667 & -69.0000 \end{bmatrix}$$

Instructor: This matrix could be used again as an exercise for Section 2.3 because it is ill-conditioned, with a condition number of 2.7585×10^5.

38. If A were invertible, then the equation $AD = I_2$ would imply that $A^{-1}AD = A^{-1}I_2$ and $D = A^{-1}$, in which case DA would be I_3. However, A cannot be invertible, because

$$DA = \begin{bmatrix} 5/9 & 2/9 & 4/9 \\ 2/9 & 8/9 & -2/9 \\ 4/9 & -2/9 & 5/9 \end{bmatrix} \neq I_3.$$

40. [M] The *stiffness matrix* is $125\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$;

$$\mathbf{f} = \begin{bmatrix} 20 \\ -10 \\ 0 \end{bmatrix} \text{ lb}$$

42. [M] $-4, 7, -13, 16$ newtons. These forces are .03 times the entries in the fourth column of D^{-1}. *Reason:* The solution of $D\mathbf{x} = (0, 0, 0, 1)$ is the fourth column of D^{-1}. Mulitply both sides of this equation by .03, to obtain $D(.03\mathbf{x}) = (0, 0, 0, .03)$. So $.03\mathbf{x}$ is the solution of $D\mathbf{u} = (0, 0, 0, .03)$.

Section 2.3, page 123

The Invertible Matrix Theorem is abbreviated here by IMT.

2. No, by the IMT. The columns are linearly dependent since they are multiples. Another reason is that the determinant is zero.

4. Yes, by the IMT. The matrix has three pivot columns.

6. No, by the IMT. The matrix is row equivalent to $\begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ and so is *not* row equivalent to I.

8. Yes, by the IMT. The matrix has 4 pivot positions.

10. No, a 4×3 matrix is not invertible.

12. **[M]** Yes. Row reduction, swapping rows 3 and 5, and swapping rows 4 and 5, produces

$$\begin{bmatrix} 5 & 4 & 3 & 6 & 3 \\ 0 & 2/5 & 4/5 & 3/5 & 4/5 \\ 0 & 0 & 1 & -21 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

3. **a.** True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

b. True. If statement (h) of the IMT is true, then so is statement (e).

c. False. Statement (g) of the IMT is true only for invertible matrices.

d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.

e. True, by the IMT. If A^T is not invertible, then statement (l) of the IMT is false, and hence statement (a) must also be false.

a. True. If statement (k) of the IMT is true, then so is statement (j).

b. True. If statement (e) of the IMT is true, then so is statement (h).

c. True. See the remark immediately following the proof of the IMT.

d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n \times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.

e. True, by the IMT. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. This fact tells us *nothing* about the columns of A.

H is invertible, by the IMT. So it is *not* possible for an equation $H\mathbf{x} = \mathbf{v}$ to have more than one solution.

No, because statement (h) of the IMT is false.

22. Statement (f) of the IMT is false for the matrix C, so (h) is false, too. Thus the columns of C do *not* span $\mathbb{R}^n$.

24. If A is invertible, then A^{-1} is invertible, by Theorem 6(a) in Section 2.2. So the IMT applies to A^{-1}, and the columns of A^{-1} are linearly independent.

26. A is invertible by the IMT, and hence A^T is invertible (by Theorem 6(c) or by the IMT). Thus the columns of A^T are linearly independent. That is, the rows of A are linearly independent.

28. One method is to let M be $(AB)^{-1}$. Then $M(AB) = I$, and $(MA)B = I$. Since B is square, this equation shows that B is invertible, by part (j) of the IMT. Another method is to show that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution and then use the IMT. If $B\mathbf{x} = \mathbf{0}$, then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. Since AB is invertible, $\mathbf{x} = \mathbf{0}$.

30. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the columns of its standard matrix A span $\mathbb{R}^n$, by Theorem 12 in Section 1.8. By the IMT, A is invertible. Hence, by Theorem 9, T is invertible.

32. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A must have a pivot in each of its n columns. Since A is square, there must be a pivot in each row of A. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and

$$T^{-1}(\mathbf{x}) = B\mathbf{x}, \text{ where } B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$$

36. Given $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, let $\mathbf{x} = S(\mathbf{u})$ and $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = T(S(\mathbf{u})) = \mathbf{u}$ and $T(\mathbf{y}) = T(S(\mathbf{v})) = \mathbf{v}$, by equation (2). Hence

$$\begin{aligned} S(\mathbf{u} + \mathbf{v}) &= S(T(\mathbf{x}) + T(\mathbf{y})) & \\ &= S(T(\mathbf{x} + \mathbf{y})) & \text{Because } T \text{ is linear} \\ &= \mathbf{x} + \mathbf{y} & \text{By equation (1)} \\ &= S(\mathbf{u}) + S(\mathbf{v}) & \end{aligned}$$

So S preserves sums. For any scalar r,

$$\begin{aligned} S(r\mathbf{u}) &= S(rT(\mathbf{x})) = S(T(r\mathbf{x})) & \text{Because } T \text{ is linear} \\ &= r\mathbf{x} & \text{By equation (1)} \\ &= rS(\mathbf{u}) & \end{aligned}$$

So S preserves scalar multiples. Thus S is a linear transformation.

38. **[M]** $\text{cond}(A) \approx 23683$, which is approximately 10^4. If you make several trials with MATLAB, which records 16 digits accurately, you should find that $\mathbf{x}$ and $\mathbf{x}_1$ agree to at least 12 or 13 significant digits. So about 4 significant digits are lost.

40. **[M]** Solve $A\mathbf{x} = (0, 0, 0, 0, 1)$. MATLAB shows that $\text{cond}(A) \approx 4.8\times 10^5$. With MATLAB, the entries in the computed value of $\mathbf{x}$ should be accurate to at least 11 digits. The exact solution is $(630, -12600, 56700, -88200, 44100)$.

Section 2.4, page 130

2. $\begin{bmatrix} AC & AD \\ BE & BF \end{bmatrix}$ **4.** $\begin{bmatrix} A - XC & B - XD \\ C & D \end{bmatrix}$

6. $X = A^{-1}$ (by the IMT, because A is square), $Z = C^{-1}$ (by the IMT, because C is square), $Y = -C^{-1}BA^{-1}$

8. $X = A^{-1}$ (by the IMT, because A is square), $Y = 0$, $Z = -A^{-1}B$

10. $X = -A + BC$, $Y = -B$, $Z = -C$

11. **a.** True. See the definition (1) in the paragraph preceding Example 4.

b. False. See Example 3. The number of columns of A_{11} and A_{12} must match the number of rows of B_1 and B_2, respectively.

12. **a.** True. See the definition (1) in the paragraph preceding Example 4.

b. False. Both BA and AB are defined, although they have different dimensions. In fact, $AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{bmatrix}$, which is the block analogue of an outer product. See Example 4.

14. The calculations in Example 5 showed that if A is invertible, then both A_{11} and A_{22} are invertible. Conversely, suppose A_{11} and A_{22} are both invertible, and define B to be the matrix that Example 5 says should be the inverse of A. A routine calculation shows that $AB = I$. Since A is square, the IMT implies that A is invertible. (Alternatively, one could also show that $BA = I$.)

16. The inverse of $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ is $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$. Similarly, $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ has an inverse. From equation (7), one obtains

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \qquad (*)$$

If A is invertible, then the matrix on the right side of $(*)$ is a product of invertible matrices and hence is invertible. By Exercise 13, A_{11} and S must be invertible.

18. The Schur complement of X^TX is $\mathbf{x}_0^T\mathbf{x}_0 - (\mathbf{x}_0^TX)(X^TX)^{-1}(X^T\mathbf{x}_0) = \mathbf{x}_0^T(I - X(X^TX)^{-1}X^T)\mathbf{x}_0 = \mathbf{x}_0^TM\mathbf{x}_0$.

20. The Schur complement of $A - BC - \lambda I$ is $I + C(A - BC - sI)^{-1}B$. *Note:* The proof that this function actually is the inverse of the $W(s)$ in Exercise 19 involves only matrix algebra, but it is a little tricky. The following algebraic identity is needed:

$$\begin{aligned} CU^{-1}B - CV^{-1}B &= C(U^{-1} - V^{-1})B \\ &= CU^{-1}(V - U)V^{-1}B \end{aligned}$$

for any invertible $n\times n$ matrices U and V and any B and C such that the multiplication is well defined.

22. **a.** $A^2 = I$, by direct calculation.

b. $M = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}$, so

$$M^2 = \begin{bmatrix} A^2 & 0 \\ 0 & (-A)^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

23. **[M]** Here are the MATLAB commands:

a. `A(15:20,5:10)`

b. `A(10:14,20:29) = B`

c. `C = A, C(21:50,31:50) = A'`

Two commands but fewer keystrokes or:

`C = [A zeros(20,20); zeros(30,30) A']`

24. **[M]** The specific commands depend on the matrix program.

c. The algebra needed comes from the block matrix equation

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

where $\mathbf{x}_1$ and $\mathbf{b}_1$ are in $\mathbb{R}^{20}$ and $\mathbf{x}_2$ and $\mathbf{b}_2$ are in $\mathbb{R}^{30}$. Then $A_{11}\mathbf{x}_1 = \mathbf{b}_1$, which can be solved to produce $\mathbf{x}_1$. The equation $A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 = \mathbf{b}_2$ yields $A_{22}\mathbf{x}_2 = \mathbf{b}_2 - A_{21}\mathbf{x}_1$, which can be solved for $\mathbf{x}_2$ by row reducing the matrix $[\, A_{22} \quad \mathbf{c}\,]$, where $\mathbf{c} = \mathbf{b}_2 - A_{21}\mathbf{x}_1$.

Section 2.5, page 139

2. $L\mathbf{y} = \mathbf{b} \Rightarrow \mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, $U\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{x} = \begin{bmatrix} 1/4 \\ 2 \\ 1 \end{bmatrix}$

4. $\mathbf{y} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

6. $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

8. $LU = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ 0 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix},$

$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},$

$U^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -1 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 6 \end{bmatrix}$, and

$A^{-1} = U^{-1}L^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -3 & -2 \\ -14 & 6 & 8 \\ -6 & 6 & 6 \end{bmatrix}$

Since L is unit lower triangular, it is invertible and may be row reduced to I by adding suitable multiples of a row to the rows below it, beginning with the top row. If elementary matrices $E_1, \dots, E_p$ implement these row operations, then

$$E_p \cdots E_1 A = (E_p \cdots E_1 L)U = IU = U$$

This shows that A may be row reduced to U using only row-replacement operations.

$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & -3 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; if $A = LU$, with only three nonzero rows in U, use the first three columns of L for B and the top three rows of U for C.

Since Q is square and $Q^TQ = I$, Q is invertible and $Q^{-1} = Q^T$, by the Invertible Matrix Theorem. Thus A is the product of invertible matrices and hence is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$. From $A\mathbf{x} = \mathbf{b}$, we have $QR\mathbf{x} = \mathbf{b}$, $Q^TQR\mathbf{x} = Q^T\mathbf{b}$, $R\mathbf{x} = Q^T\mathbf{b}$, and $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. A good algorithm for finding $\mathbf{b}$ is to compute $Q^T\mathbf{b}$ and then row reduce $[\,R \quad Q^T\mathbf{b}\,]$. (See Exercise 11 in Section 2.2.) The reduction is fast because R is triangular.

26. In general, $A^k = PD^kP^{-1}$, where $D^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix}$.

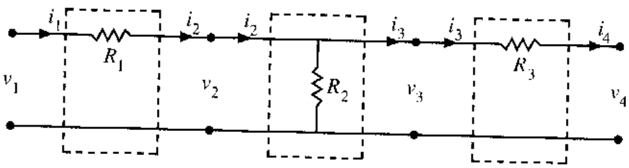
28. $\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} =$

$\begin{bmatrix} 1 & 0 \\ -(1/R_1 + 1/R_2 + 1/R_3) & 1 \end{bmatrix}$. The single shunt resistance is $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

30. The transfer matrix of the circuit below is

$$\begin{bmatrix} 1 + R_3/R_2 & -R_1 - R_3 - R_1R_3/R_2 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix}.$$

Set that matrix equal to $A = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix}$, and solve to find $R_1 = 8$ ohms, $R_2 = 4$ ohms, $R_3 = 4/3$ ohms.



32. [M] a. $L = \begin{bmatrix} 1 & & & & \\ -1/3 & 1 & & & \\ & -3/8 & 1 & & \\ & & -8/21 & 1 & \\ & & & -21/55 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 3 & -1 & & & \\ & 8/3 & -1 & & \\ & & 21/8 & -1 & \\ & & & 55/21 & -1 \\ & & & & 144/55 \end{bmatrix}$

b. Let $\mathbf{s}_k$ satisfy $L\mathbf{s}_k = \mathbf{t}_{k-1}$. Then $\mathbf{t}_k$ satisfies $U\mathbf{t}_k = \mathbf{s}_k$.

$\mathbf{t}_0 = \begin{bmatrix} 10 \\ 12 \\ 12 \\ 12 \\ 10 \end{bmatrix}; \mathbf{s}_1 = \begin{bmatrix} 10.0000 \\ 15.3333 \\ 17.7500 \\ 18.7619 \\ 17.1636 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} 6.5556 \\ 9.6667 \\ 10.4444 \\ 9.6667 \\ 6.5556 \end{bmatrix};$

$\mathbf{s}_2 = \begin{bmatrix} 6.5556 \\ 11.8519 \\ 14.8889 \\ 15.3386 \\ 12.4121 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 4.7407 \\ 7.6667 \\ 8.5926 \\ 7.6667 \\ 4.7407 \end{bmatrix}; \mathbf{s}_3 = \begin{bmatrix} 4.7407 \\ 9.2469 \\ 12.0602 \\ 12.2610 \\ 9.4222 \end{bmatrix},$

$$\mathbf{t}_3 = \begin{bmatrix} 3.5988 \\ 6.0556 \\ 6.9012 \\ 6.0556 \\ 3.5988 \end{bmatrix}; \mathbf{s}_4 = \begin{bmatrix} 3.5988 \\ 7.2551 \\ 9.6219 \\ 9.7210 \\ 7.3104 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 2.7922 \\ 4.7778 \\ 5.4856 \\ 4.7778 \\ 2.7922 \end{bmatrix}$$

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The answers here were computed by MATLAB in double precision (about 16 significant digits) and reported in the "short format" of about 5 significant digits.

2. $\mathbf{x}^{(1)} = \begin{bmatrix} 2.5 \\ 5.375 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 3.0375 \\ 5.0625 \end{bmatrix},$
$\mathbf{x}^{(3)} = \begin{bmatrix} 3.0063 \\ 4.9953 \end{bmatrix}, \ldots, \mathbf{x}^{(5)} = \begin{bmatrix} 2.9999 \\ 5.0001 \end{bmatrix}$

4. $\mathbf{x}^{(1)} = \begin{bmatrix} 2.98 \\ 1.01 \\ -1.96 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 3.0002 \\ 1.0006 \\ -2.0004 \end{bmatrix}, \mathbf{x}^{(3)} = \begin{bmatrix} 3.0000 \\ 1.0000 \\ -2.0000 \end{bmatrix}$

6. $\mathbf{x}^{(1)} = \begin{bmatrix} 2.5 \\ 5.0625 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 3.0063 \\ 4.9992 \end{bmatrix}, \ldots, \mathbf{x}^{(4)} = \begin{bmatrix} 3.0000 \\ 5.0000 \end{bmatrix}.$
Jacobi took 5 iterations for about the same accuracy.

8. $\mathbf{x}^{(1)} = \begin{bmatrix} 2.98 \\ 1.0398 \\ -2.0016 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 3.0008 \\ 1.0000 \\ -2.0000 \end{bmatrix},$
$\mathbf{x}^{(3)} = \begin{bmatrix} 3.0000 \\ 1.0000 \\ -2.0000 \end{bmatrix}$. Jacobi also took 3 iterations for about the same accuracy.

10. Only (a) is strictly diagonally dominant.

12. $\mathbf{x}^{(1)} = \begin{bmatrix} -3 \\ -22 \\ -4 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} -87 \\ -358 \\ -88 \end{bmatrix}, \mathbf{x}^{(3)} = \begin{bmatrix} -1347 \\ -5398 \\ -1348 \end{bmatrix}$. With the first two equations interchanged, Gauss–Seidel produces $\mathbf{x}^{(6)} = \begin{bmatrix} 3.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$.

14. Gauss–Seidel produces $\mathbf{x}^{(18)} = \begin{bmatrix} 10.000 \\ 13.000 \\ -15.000 \end{bmatrix}$, accurate to three decimal places. *Note:* $\mathbf{x}^{(17)} = \begin{bmatrix} 9.9991 \\ 12.9995 \\ -14.9997 \end{bmatrix}$, which rounds to three-place accuracy.

16. The Jacobi method produces a sequence of vectors whose entries oscillate, grow, and do not converge. The Gauss–Seidel method produces $\mathbf{x}^{(13)} = \begin{bmatrix} 3.9998 \\ -1.0009 \\ 2.0005 \end{bmatrix}$, accurate to within .001.

18. a. By definition,

$$M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}$$

Also, since $\mathbf{x}^*$ satisfies $A\mathbf{x}^* = \mathbf{b}$,

$$M\mathbf{x}^* = N\mathbf{x}^* + \mathbf{b}$$

Subtracting and using a property of matrix multiplication gives

$$M(\mathbf{x}^{(k+1)} - \mathbf{x}^*) = N(\mathbf{x}^{(k)} - \mathbf{x}^*)$$

Left-multiplying both sides by M^{-1}, and using the definition of the error vector $\mathbf{e}^{(k)}$, produces

$$\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} = M^{-1}N(\mathbf{x}^{(k)} - \mathbf{x}^*), \text{ and}$$
$$\mathbf{e}^{(k+1)} = M^{-1}N\mathbf{e}^{(k)}$$

b. The statement $\mathbf{e}^{(k)} = (M^{-1}N)^k\mathbf{e}^{(0)}$ is obvious for $k = 0$. Suppose it holds for some k, and apply $M^{-1}N$ to both sides. Then, using part (a), we find that

$$M^{-1}N\mathbf{e}^{(k)} = M^{-1}N(M^{-1}N)^k\mathbf{e}^{(0)}$$

and

$$\mathbf{e}^{(k+1)} = (M^{-1}N)^{k+1}\mathbf{e}^{(0)}$$

Thus, for each $k \geq 0$, the truth of the equation for k implies its truth for $k + 1$. By the principle of induction, the equation is true for all integers $k \geq 0$.

Section 2.7, page 153

Note: Exercises 2, 3, and 4 could be used for students to discover the linearity of the Leontief model.

2. $\begin{bmatrix} 33.33 \\ 35.00 \\ 15.00 \end{bmatrix}$ **4.** $\begin{bmatrix} 73.33 \\ 50.00 \\ 30.00 \end{bmatrix}$ **6.** $\begin{bmatrix} 50 \\ 45 \end{bmatrix}$

8. a. Since $\mathbf{x}$ satisfies $(I - C)\mathbf{x} = \mathbf{d}$ and $\Delta\mathbf{x}$ satisfies $(I - C)\Delta\mathbf{x} = \Delta\mathbf{d}$, linearity of matrix multiplication shows that

$$(I - C)(\mathbf{x} + \Delta\mathbf{x}) = (I - C)\mathbf{x} + (I - C)\Delta\mathbf{x} = \mathbf{d} + \Delta\mathbf{d}$$

which means that $\mathbf{x} + \Delta\mathbf{x}$ satisfies the production equation for a demand of $\mathbf{d} + \Delta\mathbf{d}$.

b. If $\Delta\mathbf{x}$ satisfies $(I - C)\Delta\mathbf{x} = \Delta\mathbf{d}$, then $\Delta\mathbf{x} = (I - C)^{-1}\Delta\mathbf{d}$, which is the first column of $(I - C)^{-1}$ in the case when $\Delta\mathbf{d}$ is the first column of I.

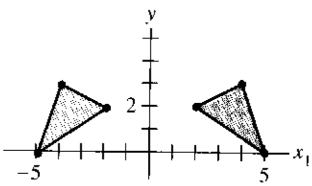
10. By the argument in Exercise 8, the effect of raising the demand for the output of one sector of the economy is given by the entries in the corresponding column of $(I - C)^{-1}$. When these entries are all positive, every sector must increase its output by some positive (though possibly small) quantity. So an increase in demand for *any* sector will increase the demand for *every* sector.

12. $D_{m+1} = I + CD_m$

14. **[M]** $x =$ (134034, 131687, 69472, 176912, 66596, 443773, 18431). In view of the remarks for Exercise 13, a realistic answer might be $\mathbf{x} = 1000 \times (134, 132, 69, 177, 67, 444, 18)$.

Section 2.8, page 163

2. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}$



4. $\begin{bmatrix} .8 & 0 & -1.6 \\ 0 & 1.2 & 3.6 \\ 0 & 0 & 1 \end{bmatrix}$ **6.** $\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 3+2\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 7-5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

10. D commutes with R but not with T; T does not commute with R.

12. Two identities: $\tan\varphi/2 = \frac{1-\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1+\cos\varphi}$. The first identity shows that $1 - (\tan\varphi/2)(\sin\varphi) = \cos\varphi$, and hence

$$\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second identity shows that

$$(\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 = -(\cos\varphi + 1)(\tan\varphi/2) = -\sin\varphi$$

Hence

$$\begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. The matrix from Exercise 7 may be written as

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3+4\sqrt{3} \\ 0 & 1 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a rotation through 60° about the origin, followed by a translation by $(3 + 4\sqrt{3}, 4 - 3\sqrt{3})$.

16. Both $(1, -2, 3, 4)$ and $(10, -20, 30, 40)$ are homogeneous coordinates for $(1/4, -1/2, 3/4)$ because of the formulas $x = X/H$, $y = Y/H$, and $z = Z/H$.

18. $\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. The triangle with vertices $(6, 2, 0)$, $(15, 10, 0)$, $(2, 3, 0)$.

22. **[M]** $\begin{bmatrix} 1.0031 & .9548 & .6179 \\ .9968 & -.2707 & -.6448 \\ 1.0085 & -1.1105 & 1.6996 \end{bmatrix}\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} R \\ G \\ B \end{bmatrix}$

Section 2.9, page 174

2. No. The set is closed under scalar multiples, but not sums. For instance, the sum of $(1, 0)$ and $(0, -1)$ is not in the set.

4. The set is closed under sums, but not under multiplication by a negative scalar.

6. No. For instructor use: $(1, -3, 11, 8)$ *is* in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

8. Yes, the augmented matrix $[\,A \quad \mathbf{u}\,]$ corresponds to a consistent system.

10. Yes, $A\mathbf{u} = \mathbf{0}$.

12. $p = 3$, $q = 4$. Nul A is a subspace of $\mathbb{R}^3$; Col A is a subspace of $\mathbb{R}^4$.

14. Nul A: $\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$, or any nonzero multiple of this vector.

Col A: any column of A

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

18. Yes. Let A be the matrix whose columns are the vectors given. Row reduction of A shows three pivots, so A is invertible and its columns form a basis for $\mathbb{R}^3$.

20. No. The vectors are linearly dependent (because there are more vectors in the set than entries in each vector), so the vectors cannot be a basis for any subspace.

22. Col A: $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$; Nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix}$

24. Col A: $\begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -3 \\ 5 \end{bmatrix}$;

Nul A: $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}$

28. The three pivot columns of A form a basis for Col A, so Col A is a 3-dimensional subspace of $\mathbb{R}^4$. The equation $A\mathbf{x} = \mathbf{0}$ has four free variables, so dim Nul $A = 4$.

30. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ **32.** $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

34. dim Col $A = 3$ **36.** rank $A = 2$

37. **a.** False. See the definition at the beginning of the section.
b. True. See the paragraph before Example 4.
c. True. See Theorem 12.
d. False. See the Warning after Theorem 13.
e. True. See Example 5.

38. **a.** False. A subspace must satisfy *three* conditions, only one of which involves the zero vector.
b. False. See the paragraph after Example 4.
c. False. See the paragraph before Example 4.
d. True. See the definition of coordinate vector.
e. True. See the paragraph after Example 11.

40. Let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans H by definition. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, it is a basis for H.

42. The p columns of A span Col A, by definition. If dim Col $A = p$, then the spanning set of p columns is automatically a basis for Col A, by the Basis Theorem, and hence the columns are linearly independent.

44. If $\mathcal{A}$ contained more vectors than $\mathcal{B}$, then $\mathcal{A}$ would be linearly dependent by Exercise 43, because $\mathcal{B}$ spans W. This is impossible, because $\mathcal{A}$ is a basis for W. Repeat the argument with $\mathcal{B}$ and $\mathcal{A}$ interchanged to conclude that $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

46. **[M]** Col A: $\begin{bmatrix} 5 \\ 4 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 6 \end{bmatrix}$;

Nul A: $\begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix}$

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18. **[M]** If J denotes the $n \times n$ matrix of ones, then

$$A_n = J - I_n \quad \text{and} \quad A_n^{-1} = \frac{1}{n-1} \cdot J - I_n$$

Proof: Observe that $J^2 = nJ$ and $A_nJ = (J - I)J = J^2 - J = (n - 1)J$. Now compute $A_n((n-1)^{-1}J - I) = (n-1)^{-1}A_nJ - A_n = J - (J - I) = I$. Since A_n is square, A_n is invertible and its inverse is $(n-1)^{-1}J - I$.

CHAPTER 3

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2. 2 **4.** 20 **6.** 1 **8.** -11

10. -6. Start with row 2.

12. 36. Start with row 1 or column 4.

14. 9. Start with row 4 or column 5.

16. 2 **18.** 20

20. $ad - bc$, $a(kd) - b(kc) = k(ad - bc)$. Scaling a row by k multiplies the determinant by k.

22. $ad - bc$, $(ad + kcd) - (bc + kdc) = ad - bc$. Row replacement does not change a determinant.

24. $2a - 6b + 3c$, $-2a + 6b - 3c$. Interchanging two rows reverses the sign of the determinant.

26. 1 **28.** k **30.** -1

k. A scaling matrix is diagonal, with k on the diagonal and with 1's as the other diagonal entries. The determinant is the product of the diagonal entries.

$\det EA = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} = akd - bkc = k(ad - bc)$

$= (\det E)(\det A)$

$\det EA = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix} = a(kb + d) - b(ka + c)$

$= akb + ad - bka - bc = (+1)(ad - bc)$

$= (\det E)(\det A)$

$\det kA = k^2 \cdot \det A$

a. True. See the paragraph preceding the definition of $\det A$.
b. False. See the definition of cofactor, preceding Theorem 1.

a. False. See Theorem 1.
b. False. See Theorem 2.

The area of the parallelogram and the determinant of $[\,\mathbf{v} \quad \mathbf{u}\,]$ are both bc. The determinant of $[\,\mathbf{u} \quad \mathbf{v}\,]$ is $-bc$. Both matrices determine the same parallelogram, with base of length c and height b.

[M] Theorem 6 in Section 3.2 will show that $\det AB = (\det A)(\det B)$.

[M] If A is invertible, then $\det A \neq 0$, by Theorem 4 in Section 3.2. Students will be asked in Exercise 31 of Section 3.2 to prove that $\det A^{-1} = 1/(\det A)$.

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A constant may be factored out of one row.

A row replacement operation does not change the determinant.

-18 **8.** 0 **10.** 24 **12.** 114

0 **16.** 21 **18.** 7 **20.** 7

Not invertible

Linearly independent

Linearly dependent

a. True. Theorem 3(a).
b. False. If scaling operations are used to produce U, then the formula described may not give $\det A$. See the paragraph following Example 2.
c. True. See the remark following Theorem 4.
d. False. See the warning after Example 5.

a. True. By Theorem 3(b), the first interchange changes only the sign of the determinant, so the second interchange restores the original sign of the determinant.
b. False. True when A is triangular (Theorem 2 in Section 3.1).
c. False. The conditions described provide only some cases when $\det A$ is zero. See the paragraph after Theorem 4.
d. False. See Theorem 5.

30. If two rows are equal, interchange them. This doesn't change the matrix, but the sign of the determinant is reversed. This is possible only if the determinant is zero. The result about columns can be explained the same way, or one can remark that if A has two equal columns, then A^T has two equal rows. In this case, $\det A^T = 0$. So $\det A = 0$, too, by Theorem 5.

32. $\det(rA) = r^n \cdot \det A$

34.

$\det(PAP^{-1})$	$= (\det P)(\det A)(\det P^{-1})$	By Theorem 6
	$= (\det P)(\det A)(\det P)^{-1}$	By Exercise 31
	$= \det A$	

36. $0 = \det A^4 = (\det A)^4$, by Theorem 6. So $\det A = 0$, which implies that A is not invertible, by Theorem 4.

38. $\det AB = \det \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix} = 0$

$(\det A)(\det B) = (-6 + 6)(-4 + 2) = 0$

40. **a.** -2 **b.** 32 **c.** -16 **d.** 1 **e.** -1

42. $\det(A + B) = \det \begin{bmatrix} 1 + a & b \\ c & 1 + d \end{bmatrix} =$

$1 + a + d + ad - bc$. Also $\det A + \det B = 1 + (ad - bc)$. Since $\det(A + B) - (\det A + \det B) = a + d$, we have $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

44.

$\det AE$	$= \det(AE)^T$	Theorem 5
	$= \det E^TA^T$	Section 2.1
	$= (\det E^T)(\det A^T)$	Theorem 6
	$= (\det E)(\det A)$	Theorem 5 used twice

46. **[M]** For A as in Exercise 11 of Section 2.3, $\det A = -1$ and $\operatorname{cond} A = 23683$. Although A is nearly singular, it has an inverse:

$$A^{-1} = \begin{bmatrix} 0 & -2 & 1 & -1 \\ -14 & -401 & 195 & -203 \\ 19 & 549 & -267 & -278 \\ -7 & -196 & 95 & -99 \end{bmatrix}$$

The determinant is sensitive to scaling, but the condition number does not change:

$\det(10A) = 10^4(-1)$, $\det(.1A) = 10^{-4}(-1)$, but

$\operatorname{cond}(10A) = \operatorname{cond}(.1A) = \operatorname{cond} A$.

The same things happen when $A = I_4$.

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2. $\begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$ **4.** $\begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ **6.** $\begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$

8. All real s; $x_1 = \dfrac{3s+2}{3(s^2+3)}$, $x_2 = \dfrac{2s-9}{5(s^2+3)}$

10. $s \neq 0, 1/4$; $x_1 = \dfrac{6s-2}{3s(4s-1)}$, $x_2 = \dfrac{1}{3(4s-1)}$

12. $\operatorname{adj} A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}, A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$

14. $\operatorname{adj} A = \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}, A^{-1} = (-1)\begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$

16. $\operatorname{adj} A = \begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}, A^{-1} = -\frac{1}{9}\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

18. Each cofactor in A is an integer because it is just a sum of products of entries of A. Hence all the entries in adj A are integers. Since $\det A = 1$, the inverse formula in Theorem 8 shows that all the entries in A^{-1} are integers.

20. 7 **22.** 21 **24.** 15

26. By definition, $\mathbf{p} + S$ is the set of all vectors of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is in S. Applying T to a typical vector in $\mathbf{p} + S$, we have $T(\mathbf{p} + \mathbf{v}) = T(\mathbf{p}) + T(\mathbf{v})$. This vector is in the set denoted by $T(\mathbf{p}) + T(S)$. This proves that T maps the set $\mathbf{p} + S$ into the set $T(\mathbf{p}) + T(S)$.

Conversely, any vector in $T(\mathbf{p}) + T(S)$ has the form $T(\mathbf{p}) + T(\mathbf{v})$ for some $\mathbf{v}$ in S. This vector may be written as $T(\mathbf{p} + \mathbf{v})$. This shows that every vector in $T(\mathbf{p}) + T(S)$ is the image under T of some point in $\mathbf{p} + S$.

28. Use Theorem 8. Or, compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 14 & 2 \\ -3 & 1 \end{bmatrix}$$

The determinant of this matrix is 20.

30. Let $\mathbf{p} = (x_3, y_3)$ and let $R' = R - \mathbf{p}$. The vertices of R' are $\mathbf{v}_1 = (x_1 - x_3, y_1 - y_3)$, $\mathbf{v}_2 = (x_2 - x_3, y_2 - y_3)$, and the origin. Then

$$\begin{aligned}\{\text{area of } R\} &= \{\text{area of } R'\} \\ &= \frac{1}{2}\left\{\begin{array}{c}\text{area of parallelogram} \\ \text{determined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{array}\right\} \\ &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\right| \qquad (1)\end{aligned}$$

Also, using row operations, we get

$$\begin{aligned}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} &= \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix} \\ &= \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} \\ &= \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\end{aligned}$$

This calculation and (1) give the desired result.

32. From the formula in the exercise,

$$\{\text{volume of } S\} = \tfrac{1}{3}\{\text{area of base}\}\cdot\{\text{height}\} = \tfrac{1}{6}$$

because the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ have unit length. The tetrahedron S' with vertices at $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is the image of S under the linear transformation T such that $T(\mathbf{e}_1) = \mathbf{v}_1$, $T(\mathbf{e}_2) = \mathbf{v}_2$, and $T(\mathbf{e}_3) = \mathbf{v}_3$. The standard matrix for T is $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]$. By Theorem 10,

$$\{\text{volume of } S'\} = |\det A|\cdot\tfrac{1}{6} = \tfrac{1}{6}|\det[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]|$$

34. **[M]** MATLAB:

```
x2 = det([A(:,1)  b  A(:,3:4)])/det(A)
```

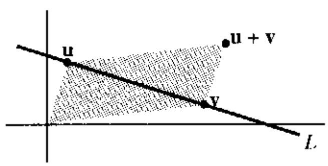
CHAPTER 4

Section 4.1, page 217

2. **a.** Given $\begin{bmatrix} x \\ y \end{bmatrix}$ in W and any scalar c, the vector $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$, since $xy \geq 0$.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u} + \mathbf{v}$ is not in W.

4.



u and **v** are on the line, but **u** + **v** is not.

6. No, the zero polynomial is not in the set.

8. Yes. The zero vector is in the set, H. If $\mathbf{p}$ and $\mathbf{q}$ are in H, then $(\mathbf{p} + \mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0$, so $\mathbf{p} + \mathbf{q}$ is in H. Also, for any scalar c, $(c\mathbf{p})(0) = c\cdot\mathbf{p}(0) = c\cdot 0 = 0$, so $c\mathbf{p}$ is in H.

. $H = \mathrm{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. By Theorem 1, H is a subspace of $\mathbb{R}^3$.

. $W = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. By Theorem 1, W is a subspace of $\mathbb{R}^4$.

. No, because the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$ has no solution, as revealed by an echelon form of the augmented matrix for this equation.

. Not a vector space because the zero vector is not in W

. $S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

. **a.** The constant function $\mathbf{f}(t) = 0$ is continuous. The sum of two continuous functions is continuous. A constant multiple of a continuous function is continuous.

. Yes. See the proof of Theorem 12 in Section 2.9 for a proof that is similar to the one needed here.

. **a.** False. The zero vector in V is the function $\mathbf{f}$ whose values $\mathbf{f}(t)$ are zero *for all* t in $\mathbb{R}$. See Example 5.
b. False. See the definition of a vector. An arrow in three-dimensional space is an example of a vector, but not every vector is such an arrow.
c. False. Exercises 1, 2, and 3 each provide an example of a subset that contains the zero vector but is not a subspace.
d. True. See the paragraph before Example 6.
e. False. Digital signals are used. See Example 3.

a. True. See the definition of a vector space.
b. True. See statement (3) in the box before Example 1.
c. True. See the paragraph before Example 6.
d. False. See Example 8.
e. False. The second and third parts of the conditions are stated incorrectly. In part (ii) here, for example, there is no statement that $\mathbf{u}$ and $\mathbf{v}$ represent all possible elements of H.

a. 3 **b.** 5 **c.** 4

a. 4 **b.** 7 **c.** 3 **d.** 5 **e.** 4

$\mathbf{u} = 1 \cdot \mathbf{u}$	Axiom 10
$= c^{-1}c \cdot \mathbf{u} = c^{-1}(c\mathbf{u})$	Axiom 9
$= c^{-1}\mathbf{0} = \mathbf{0}$	Property (2)

32. Both H and K contain the zero vector of V because they are subspaces of V. Hence $\mathbf{0}$ is in $H \cap K$. Take $\mathbf{u}$ and $\mathbf{v}$ in $H \cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in both H and K. Since H is a subspace, $\mathbf{u} + \mathbf{v}$ is in H. Likewise, $\mathbf{u} + \mathbf{v}$ is in K. Hence $\mathbf{u} + \mathbf{v}$ is in $H \cap K$. For any scalar c, the vector $c\mathbf{u}$ is in both H and K because they are subspaces. Hence $c\mathbf{u}$ is in $H \cap K$. Thus $H \cap K$ is a subspace.

The union of two subspaces is not, in general, a subspace. In $\mathbb{R}^2$, let H be the x-axis and K the y-axis. The sum of a nonzero vector in H and a nonzero vector in K is not on either the x-axis or the y-axis. So $H \cup K$ is not closed under vector addition, and $H \cup K$ is not a subspace of $\mathbb{R}^2$.

34. **[M]** An echelon form of $[\,A \quad \mathbf{y}\,]$ shows that $A\mathbf{x} = \mathbf{y}$ is consistent. In fact, $\mathbf{x} = (5.5, -2, 3.5)$.

36. **[M]** The functions are $\sin 3t$, $\cos 4t$, and $\sin 5t$.

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2. $\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\mathbf{w}$ is in Nul A.

4. $\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

8. W is not a subspace because $\mathbf{0}$ is not in W. The vector $(0, 0, 0)$ does not satisfy the condition $5r - 1 = s + 2t$.

10. W is a subspace of $\mathbb{R}^4$ because W is the set of solutions of the system

$$\begin{aligned} a + 3b - c \phantom{{}-d} &= 0 \\ a + b + c - d &= 0 \end{aligned}$$

12. If $(b - 5d, 2b, 2d + 1, d)$ were the zero vector, then $2d + 1 = 0$ and $d = 0$, which is impossible. So $\mathbf{0}$ is not in W, and W is not a subspace.

14. $W = \mathrm{Col}\, A$ for $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$, so W is a vector space by Theorem 3.

16. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

18. **a.** 3 **b.** 4 **20.** **a.** 5 **b.** 1

22. $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ in Nul A, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Col A. Other answers possible.

24. $\mathbf{w}$ is in both Nul A and Col A.
$A\mathbf{w} = \mathbf{0}$, and $\mathbf{w} = -\frac{1}{2}\mathbf{a}_1 + \mathbf{a}_2$

25. **a.** True, by the definition before Example 1.
b. False. See Theorem 2.
c. True. See the remark just before Example 4.
d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on p. 226.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the remark after Theorem 3.

26. **a.** True. See Theorem 2. (A subspace is itself a vector space.)
b. True. See Theorem 3.
c. False. See the box after Theorem 3.
d. True. See the paragraph after the definition of a linear transformation.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the paragraph before Example 8.

28. The two systems have the form $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = 5\mathbf{v}$. Since the first system is consistent, $\mathbf{v}$ is in Col A. Since Col A is a subspace of $\mathbb{R}^3$, $5\mathbf{v}$ is also in Col A. Thus the second system is consistent.

30. The zero vector $\mathbf{0}_W$ of W is in the range of T, because the linear transformation maps the zero vector of V to $\mathbf{0}_W$. Typical vectors in the range of T are $T(\mathbf{x})$ and $T(\mathbf{w})$, where $\mathbf{x}, \mathbf{w}$ are in V. Since T is a linear transformation,

$$T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w}) \quad \text{In the range of } T$$

Thus the range of T is closed under vector addition. Also, for any scalar c, $c \cdot T(\mathbf{x}) = T(c\mathbf{x})$, since T is a linear transformation. Thus $c \cdot T(\mathbf{x})$ is in the range of T, so the range is closed under scalar multiplication. Hence the range of T is a subspace of W.

32. $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$. The range of T is $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real} \right\}$.

34. The kernel of T is $\{\mathbf{0}\}$.

36. Since Z is a subspace of W, the zero vector $\mathbf{0}_W$ of W is in Z. Because T is linear, T maps the zero vector $\mathbf{0}_V$ of V to $\mathbf{0}_W$. Thus $\mathbf{0}_V$ is in $U = \{\mathbf{x} : T(\mathbf{x}) \text{ is in } Z\}$. Now take $\mathbf{u}_1, \mathbf{u}_2$ in U. Since T is linear,

$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) \qquad (*)$$

By definition of Z, $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are in Z, and so the sum on the right of (*) is in Z because Z is a subspace. This proves that $\mathbf{u}_1 + \mathbf{u}_2$ is in U, so U is closed under vector addition. For any scalar c, $c \cdot T(\mathbf{u}_1)$ is in Z because Z is a subspace. Since T is linear, $T(c\mathbf{u}_1)$ is in Z. Hence $c\mathbf{u}_1$ is in U. Thus U is a subspace of V.

37. **[M]** $\mathbf{w}$ is in Col A. In fact, $\mathbf{w} = A\mathbf{x}$ for

$\mathbf{x} = (1/95, -20/19, -172/95, 0)$.

$\mathbf{w}$ is not in Nul A because $A\mathbf{w} = (14, 0, 0, 0)$.

38. **[M]** $\mathbf{w}$ is in Col A and in Nul A because $\mathbf{w} = A\mathbf{x}$ for $\mathbf{x} = (-2, 3, 0, 1)$, and $A\mathbf{w} = (0, 0, 0, 0)$.

39. **[M]** The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. Most students will row reduce $[\,B \quad \mathbf{a}_3\,]$ and $[\,B \quad \mathbf{a}_5\,]$ to show that the equations $B\mathbf{x} = \mathbf{a}_3$ and $B\mathbf{x} = \mathbf{a}_5$ are consistent. You can use a discussion of this part to lead into Examples 8 and 9 in Section 4.3.
b. The method of Example 3 produces $(-1/3, -1/3, 1, 0, 0)$ and $(-10/3, 26/3, 0, 4, 1)$.
c. This part reviews Section 1.8. An echelon form of A shows that the columns of A are linearly dependent and do not span $\mathbb{R}^4$. By Theorem 12 in Section 1.8, T is not one-to-one and T does not map $\mathbb{R}^5$ onto $\mathbb{R}^4$.

40. **[M]** Row reduction of $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad -\mathbf{v}_3 \quad -\mathbf{v}_4 \quad \mathbf{0}\,]$ yields

$$\begin{bmatrix} 1 & 0 & 0 & -10/3 & 0 \\ 0 & 1 & 0 & 26/3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

The general solution is a multiple of $(10, -26, 12, 3)$. One choice for $\mathbf{w}$ is $10\mathbf{v}_1 - 26\mathbf{v}_2$ ($= 12\mathbf{v}_3 + 3\mathbf{v}_4$), which is $(24, -48, -24)$. Another choice is $\mathbf{w} = (1, -2, -1)$.

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2. No, the set is linearly dependent because the zero vector is in the set. The columns of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ do not span $\mathbb{R}^3$, by the Invertible Matrix Theorem.

4. Yes. See Example 5 for an example of a justification.

6. No. $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. The matrix does not have a pivot in each row, so its columns do not span $\mathbb{R}^3$ and hence do not form a basis. However, the columns are linearly independent because they are not multiples. (More precisely, neither column is a multiple of the other.)

8. No, the vectors are linearly dependent because there are more vectors than entries in each vector. However, the vectors do span $\mathbb{R}^3$.

10. $\begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

14. Basis for Nul A: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}$

6. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

8. [M] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

0. The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ or $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

1. a. False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.
   b. False. The set $\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.
   c. True. See Example 3.
   d. False. See the subsection "Two Views of a Basis."
   e. False. See the box before Example 9.

. a. False. The subspace spanned by the set must also coincide with H. See the definition of a basis.
   b. True, by the Spanning Set Theorem, applied to V instead of H. (V is nonzero because the spanning set uses nonzero vectors.)
   c. True. See the subsection "Two Views of a Basis."
   d. False. See two paragraphs before Example 8.
   e. False. See the warning after Theorem 6.

Let $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]$. Since A is square and its columns are linearly independent, its columns also span $\mathbb{R}^n$, by the Invertible Matrix Theorem. So $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

A basis is $\{\sin t, \sin 2t\}$ because this set is linearly independent (by inspection), and $\sin t \cos t = \frac{1}{2}\sin 2t$, as pointed out in Example 2.

$\{e^{-bt}, te^{-bt}\}$. The set is linearly independent because neither function is a *scalar* multiple of the other, and the set spans H.

There are more vectors than there are entries in each vector. By Theorem 8 in Section 1.6, the set is linearly dependent and therefore cannot be a basis for $\mathbb{R}^n$.

Suppose that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist $c_1, \dots, c_p$, not all zero, such that

$$c_1 T(\mathbf{v}_1) + \cdots + c_p T(\mathbf{v}_p) = \mathbf{0}$$

Since T is linear and $\mathbf{0} = T(\mathbf{0})$,

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = T(\mathbf{0})$$

By hypothesis, T is one-to-one, so this equation implies that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent.

33. [M] For instance, writing (5) with $t = 0, .1, .2$, and $.3$, the coefficient matrix of the homogeneous system $A\mathbf{c} = \mathbf{0}$ is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ .1 & .0998 & .9801 & .0993 \\ .2 & .1987 & .9211 & .1947 \\ .3 & .2955 & .8253 & .2823 \end{bmatrix} \quad \text{(to four decimal places)}$$

The matrix A is invertible, so the equation $A\mathbf{c} = \mathbf{0}$ has only the trivial solution. Thus $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions. The MATLAB commands to create A are

```
t = [0 .1 .2 .3]'
A = [t sin(t) cos(2*t) sin(t).*cos(t)]
```

Notice the `.*` symbol, which multiplies matrices entrywise.

34. [M] The MATLAB commands

```
t = [0 .1 .2 .3 .4 .5 .6]'
A = [t  cos(t)  cos(t).^2  cos(t).^3
     cos(t).^4  cos(t).^5 cos(t).^6]
```

produce the coefficient matrix A for a 7×7 system of equations of the form

$$c_1 + c_2 \cdot \cos t + c_3 \cdot (\cos t)^2 + \cdots + c_7 \cdot (\cos t)^6 = 0$$

The matrix A is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution, which shows that the functions $1, \cos t, \dots, \cos^6 t$ are linearly independent.

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2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$ **8.** $\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$ **12.** $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ **14.** $\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$

15. a. True, by definition of the $\mathcal{B}$-coordinate vector.
   b. False. See equation (4).
   c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

16. a. True. See Example 2.

b. False. By definition, the coordinate mapping goes in the reverse direction.

c. True, when the plane passes through the origin, as in Example 7.

18. Since $\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \cdots + 0 \cdot \mathbf{b}_n$, the $\mathcal{B}$-coordinate vector of $\mathbf{b}_1$ is

$$[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1$$

For each k, $\mathbf{b}_k = 0 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 0 \cdot \mathbf{b}_n$, so $[\mathbf{b}_k]_{\mathcal{B}} = (0, \ldots, 1, \ldots, 0) = \mathbf{e}_k$.

20. For $\mathbf{w}$ in V, there exist scalars $k_1, \ldots, k_4$ such that

$$\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_4\mathbf{v}_4 \qquad (1)$$

because $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ spans V. Also, because the set is linearly dependent, there exist scalars $c_1, \ldots, c_4$, not all zero, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4$$

Adding gives

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_4 + c_4)\mathbf{v}_4$$

At least one of the weights here differs from the corresponding weight in (1) because at least one of the c_i is nonzero. So $\mathbf{w}$ is expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_4$.

22. Let $P_{\mathcal{B}} = [\,\mathbf{b}_1 \ \cdots \ \mathbf{b}_n\,]$. Then $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$. As mentioned in the text, the correspondence $\mathbf{x} \mapsto P_{\mathcal{B}}^{-1}\mathbf{x}$ is the coordinate mapping, so the desired matrix is $A = P_{\mathcal{B}}^{-1}$.

24. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_{\mathcal{B}} = \mathbf{y}$. So the coordinate mapping transforms $\mathbf{u}$ into $\mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto.

26. $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if there exist scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \qquad (2)$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} \qquad (3)$$

Conversely, (2) implies (3) because the coordinate mapping is one-to-one. Thus $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if (3) holds for some $c_1, \ldots, c_p$, which is equivalent to saying that $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$.

28. Linearly dependent because the coordinate vectors $\begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$ are linearly dependent.

30. Linearly dependent. The coordinate vectors $\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -12 \\ 9 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix}$ are linearly dependent.

32. **[M]** Linearly dependent. The coordinate vectors $\begin{bmatrix} 5 \\ -3 \\ 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 1 \\ 8 \\ -6 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \\ 5 \\ 0 \end{bmatrix}$ are linearly dependent.

34. **[M]** Row reduction of $[\,\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3\,]$ shows that there is a pivot in each column, so the columns are linearly independent and hence form a basis for the subspace H which they span.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

36. **[M]** $\begin{bmatrix} 1.30 \\ .75 \\ 1.60 \end{bmatrix}$

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2. $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$; dim is 2

4. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$; dim is 2

6. $\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$; dim is 2

$\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$; dim is 3

2 **12.** 3 **14.** 3, 3 **16.** 0, 2 **18.** 1, 2

a. True. See the box before Example 5.
b. False, unless the plane is through the origin. Read Example 4 carefully.
c. False. The dimension is 5. See Example 1.
d. False. S must have exactly n elements to be a basis for V. See Theorem 10.
e. True. See Practice Problem 2.

a. False. The only subspaces of $\mathbb{R}^3$ are listed in Example 4. $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$, because vectors in $\mathbb{R}^3$ have three coordinates. Review Example 8 in Section 4.1.
b. False. The number of *free* variables equals the dimension of Nul A. See the box before Example 5.
c. False. Read carefully the definition before Example 1. *Not* being spanned by a finite set is not the same as being spanned by an infinite set. The space $\mathbb{R}^2$ is finite-dimensional, yet it is spanned by the infinite set S of all vectors of the form (x, y), where x and y are integers. (Of course, the two vectors $(1, 0)$ and $(0, 1)$ in S by themselves span $\mathbb{R}^2$.)
d. False. S must have exactly n elements to be a basis of V. See the Basis Theorem.
e. True. See Example 4.

Obviously, none of the Laguerre polynomials is a linear combination of the Laguerre polynomials of lower degree. By Theorem 4 (Section 4.3), the set of polynomials is linearly independent. Since this set contains four vectors, and $\mathbb{P}_3$ is four-dimensional, the set is a basis of $\mathbb{P}_3$, by the Basis Theorem.

$[\mathbf{p}]_{\mathcal{B}} = (5, -4, 3)$

If dim $V = 0$, the statement is obvious. Otherwise, H contains a basis, consisting of n linearly independent vectors. By the Basis Theorem applied to V, the vectors form a basis for V.

The space $C(\mathbb{R})$ contains the space $\mathbb{P}$ as a subspace. If $C(\mathbb{R})$ were finite-dimensional, $\mathbb{P}$ would be finite-dimensional, too, by Theorem 11. This is not true, by Exercise 27, so $C(\mathbb{R})$ is infinite-dimensional.

. False. This is *not* Theorem 9. If $\mathbf{x}$ in V is nonzero, the set $\{\mathbf{0}, \mathbf{x}, 2\mathbf{x}, \dots, (p-1)\mathbf{x}\}$ is linearly dependent, no matter what the dimension of V.
. True. If dim V were less than or equal to p, V would have a basis of not more than p elements. Such a set would span V. Since this is not the case, dim V must be greater than p.
c. False. *Counterexample:* Take any nonzero vector $\mathbf{v}$, and consider the set $\{\mathbf{v}, 2\mathbf{v}, 3\mathbf{v}, \dots, (p-1)\mathbf{v}\}$.

32. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be a basis for H. Then $\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_p)\}$ spans $T(H)$, as is easily seen. Further, since T is one-to-one, Exercise 32 in Section 4.3 shows that $\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_p)\}$ is linearly independent. So this set of images is a basis for $T(H)$. So dim $H = p$ and dim $T(H) = p$.

33. **[M]** **a.** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$

b. The first k columns of A are pivot columns because, by assumption, the original k vectors are linearly independent. Col $A = \mathbb{R}^n$, because the columns of A include all the columns of the identity matrix.

34. **[M]** The $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$ are the columns of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ & 1 & 0 & -3 & 0 & 5 & 0 \\ & & 2 & 0 & -8 & 0 & 18 \\ & & & 4 & 0 & -20 & 0 \\ & & & & 8 & 0 & -48 \\ & & & & & 16 & 0 \\ & & & & & & 32 \end{bmatrix}$$

a. This problem is an **[M]** exercise because it involves a large matrix. However, one should always think about a problem before rushing to use a matrix program. Actually, neither part of this exercise requires a matrix program. Simply observe that the matrix P is invertible because it is triangular with nonzero entries on the diagonal. So the columns of P are linearly independent. Because the coordinate mapping is an isomorphism, the vectors in $\mathcal{C}$ are linearly independent.
b. dim $H = 7$, because $\mathcal{B}$ is a basis for H with 7 elements. Since $\mathcal{C}$ is linearly independent, and the vectors in $\mathcal{C}$ lie in H (because of the trig identities), $\mathcal{C}$ is a basis for H, by the Basis Theorem. (Another argument is to use the fact that the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$ span $\mathbb{R}^7$, so the vectors in $\mathcal{C}$ span H, but you must distinguish between vectors in $\mathbb{R}^7$ and vectors in H.)

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2. rank $A = 3$; dim Nul $A = 2$;

Basis for Col A: $\begin{bmatrix}1\\2\\-3\\3\end{bmatrix}, \begin{bmatrix}4\\-6\\-6\\4\end{bmatrix}, \begin{bmatrix}9\\-10\\-3\\0\end{bmatrix}$

Row A: $(1, -3, 0, 5, -7)$, $(0, 0, 2, -3, 8)$, $(0, 0, 0, 0, 5)$

Basis for Nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

4. rank $A = 3$; dim Nul $A = 3$;

Basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix}$

Basis for Row A: $(1, 1, -3, 7, 9, -9)$, $(0, 1, -1, 3, 4, -3)$, $(0, 0, 0, 1, -1, -2)$

Basis for Nul A: $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

6. 0, 3, 3

8. 2. It is impossible for Col A to be $\mathbb{R}^4$ because the vectors in Col A have 5 entries. Col A is a four-dimensional subspace of $\mathbb{R}^5$.

10. 1 **12.** 2

14. 3, 3. If A is 4×3, its rows are in $\mathbb{R}^3$ and there can be at most three linearly independent vectors in such a set. If A is 4×3, it cannot have more than three linearly independent rows because there are only three rows.

16. 0

17. **a.** True. The row vectors in A are identified with the columns of A^T. See the paragraph before Example 1.
b. False. See the warning after Example 2.
c. True. See the Rank Theorem.
d. False. See the Rank Theorem. The sum of the two dimensions equals the number of *columns* in A.
e. True. See the Numerical Note before the Practice Problem.

18. **a.** False. Review the warning after the proof of Theorem 6 in Section 4.3.
b. False. See the warning after Example 2. For instance, a row interchange usually changes dependence relations among the rows.
c. True. See the remark in the proof of the Rank Theorem.
d. True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of a matrix—say, A^T—are the columns of its transpose, and $A^{TT} = A$.
e. True. See Theorem 13.

20. No. The presence of two free variables indicates that the null space of the coefficient matrix A is two-dimensional. Since there are eight unknowns, A has eight columns and therefore must have rank 6, by the Rank Theorem. Since there are only six equations, A has six rows, and Col A is a subspace of $\mathbb{R}^6$. Since rank $A = 6$, we conclude that Col $A = \mathbb{R}^6$, which means that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

22. No. The coefficient matrix A is 10×12 and hence has rank at most 10. By the Rank Theorem, dim Nul A will be at least 2, so Nul A cannot be spanned by one vector.

24. The coefficient matrix A in this case is 7×6. It is possible that for some $\mathbf{b}$ in $\mathbb{R}^7$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. In this case, there are no free variables, so the rank of A must equal the number of columns, by the Rank Theorem. However, in any case, the rank of A cannot exceed 6, and so Col A must be a proper subspace of $\mathbb{R}^7$. Thus there exist vectors in $\mathbb{R}^7$ that are not in Col A. For such right-hand sides, the equation $A\mathbf{x} = \mathbf{b}$ will have no solution.

26. When an $m\times n$ matrix A has more rows than columns, A can have at most n pivot columns. So A has *full* rank when all n columns are pivot columns. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, that is, if and only if the columns of A are linearly independent.

28. **a.** dim Row A = dim Col A = rank A, by the Rank Theorem. So part (a) follows from the second part of that theorem.
b. Apply part (a) with A replaced by A^T and use the fact that Row A^T is just Col A.

30. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if

$$\text{rank}\,[\,A \quad \mathbf{b}\,] = \text{rank}\,A$$

because the two ranks are equal if and only if $\mathbf{b}$ is not a pivot column of $[\,A \quad \mathbf{b}\,]$. The result follows now from Theorem 2 in Section 1.2.

32. $\mathbf{v} = (1, -3, 4) = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

34. Since A can be reduced to an echelon form U by row operations, there exist invertible elementary matrices $E_1, \dots, E_p$, such that $(E_p \cdots E_1)A = U$, and $A = (E_p \cdots E_1)^{-1}U$, since the product of invertible matrices is invertible. Let $E = (E_p \cdots E_1)^{-1}$. Then $A = EU$. Denote the columns of E by $\mathbf{c}_1, \dots, \mathbf{c}_m$. Since rank $A = r$, its echelon form U has r nonzero rows, which we can denote by $\mathbf{d}_1^T, \dots, \mathbf{d}_r^T$. By the column–row expansion of EU

(Theorem 10 in Section 2.4),

$$A = EU = [\,\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_m\,]\begin{bmatrix} \mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_r^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \cdots + \mathbf{c}_r\mathbf{d}_r^T$$

[M] **a.** Many answers are possible. Here are the "canonical" choices, for $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_7\,]$:

$$C = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_4 \quad \mathbf{a}_6\,], \quad N = \begin{bmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

b. $M = [\,2 \quad 41 \quad 0 \quad -28 \quad 11\,]^T$. The matrix $[\,R^T \quad N\,]$ is 7×7 because the columns of R^T and N are in $\mathbb{R}^7$, and $\dim \operatorname{Row} A + \dim \operatorname{Nul} A = 7$. The matrix $[\,C \quad M\,]$ is 5×5 because the columns of C and M are in $\mathbb{R}^5$ and $\dim \operatorname{Col} A + \dim \operatorname{Nul} A^T = 5$, by Exercise 28(b). The invertibility of these matrices follows from the fact that their columns are linearly independent, which can be proved from Theorem 3 in Section 6.1.

M] In most cases, C will be 6×4, constructed from the rst four columns of A, R will be 4×7, N will be 7×3, and I will be 6×2.

M] The C and R given for Exercise 35 work here, and $= CR$.

M] In general, if A is nonzero, then $A = CR$ because

$$R = C[\,\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n\,] = [\,C\mathbf{r}_1 \quad C\mathbf{r}_2 \quad \cdots \quad C\mathbf{r}_n\,]$$

explain why the matrix on the right is A itself, we nsider the pivot columns of A (i.e., the columns of C) and en consider the nonpivot columns of A.

The ith pivot column of R is $\mathbf{e}_i$ (the ith column of the entity matrix). So $C\mathbf{e}_i$ is the ith pivot column of A. Since A R have pivot columns in the same location, when C mullies a pivot column of R, the result is a pivot column of A, he correct location.

A nonpivot column of R—say, $\mathbf{r}_j$—contains the weights needed to construct column j of A from the pivot columns in A, as discussed in Example 9 of Section 4.3 and the paragraph preceding that example. Thus $\mathbf{r}_j$ contains the weights needed to construct column j from the columns of C, so $C\mathbf{r}_j = \mathbf{a}_j$.

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2. a. $\begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 10 \\ 11 \end{bmatrix}$ **4.** (i)

6. a. $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

8. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

10. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$

11. a. False. See Theorem 15.

b. True. See the first paragraph in the subsection "Change of Basis in $\mathbb{R}^n$."

12. a. True. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are coordinate vectors of the linearly independent set $\mathcal{B}$. See the second paragraph after Theorem 15.

b. False. The row reduction is discussed after Example 2. The matrix P obtained there satisfies $[\mathbf{x}]_\mathcal{C} = P[\mathbf{x}]_\mathcal{B}$.

14. a. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$

b. Solve $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and obtain

$$t^2 = 3(1 - 3t^2) - 2(2 + t - 5t^2) + (1 + 2t).$$

16. a. $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\cdot\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. *Reason:* $[\mathbf{b}_j]_\mathcal{D} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_j]_\mathcal{C}$. So, by Theorem 15 and the definition of matrix multiplication:

$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \big[\,[\mathbf{b}_1]_\mathcal{D} \quad [\mathbf{b}_2]_\mathcal{D}\,\big] = \big[\,\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_1]_\mathcal{C} \quad \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{b}_2]_\mathcal{C}\,\big]$$
$$= \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\big[\,[\mathbf{b}_1]_\mathcal{C} \quad [\mathbf{b}_2]_\mathcal{C}\,\big] = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\cdot\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$

17. a. [M] $P^{-1} =$

$$\frac{1}{32}\begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ & 32 & 0 & 24 & 0 & 20 & 0 \\ & & 16 & 0 & 16 & 0 & 15 \\ & & & 8 & 0 & 10 & 0 \\ & & & & 4 & 0 & 6 \\ & & & & & 2 & 0 \\ & & & & & & 1 \end{bmatrix}$$

b. $\cos^2 t = (1/2)[1 + \cos 2t]$
$\cos^3 t = (1/4)[3\cos t + \cos 3t]$
$\cos^4 t = (1/8)[3 + 4\cos 2t + \cos 4t]$
$\cos^5 t = (1/16)[10\cos t + 5\cos 3t + \cos 5t]$
$\cos^6 t = (1/32)[10 + 15\cos 2t + 6\cos 4t + \cos 6t]$

18. **[M]** Let $C = \{\mathbf{y}_0, \dots, \mathbf{y}_6\}$, where $\mathbf{y}_k$ is the function $\cos kt$. Then the C-coordinate vector of $5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t$ is $(0, 0, 0, 5, -6, 5, -12)$. Left-multiplication by the inverse of the matrix P in Exercise 17 changes this C-coordinate vector into the $\mathcal{B}$-coordinate vector $(-6, 55/8, -69/8, 45/16, -3, 5/16, -3/8)$. So the integral (8) in this exercise equals

$$\int \left[-6 + \frac{55}{8}\cos t - \frac{69}{8}\cos 2t + \frac{45}{16}\cos 3t - 3\cos 4t + \frac{5}{16}\cos 5t - \frac{3}{8}\cos 6t\right] dt$$

From calculus, the integral equals

$$-6t + \frac{55}{8}\sin t - \frac{69}{16}\sin 2t + \frac{15}{16}\sin 3t - \frac{3}{4}\sin 4t + \frac{1}{16}\sin 5t - \frac{1}{16}\sin 6t + C$$

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2. If $y_k = 3^k$, then $y_{k+1} = 3^{k+1}$ and $y_{k+2} = 3^{k+2}$. Substituting these formulas into the left side of the equation gives

$$\begin{aligned} y_{k+2} - 9y_k &= 3^{k+2} - 9 \cdot 3^k \\ &= 3^k(3^2 - 9) = 3^k(0) = 0 \quad \text{for all } k \end{aligned}$$

Since the difference equation holds for all k, 3^k is a solution. A similar calculation works for $y_k = (-3)^k$:

$$\begin{aligned} y_{k+2} - 9y_k &= (-3)^{k+2} - 9(-3)^k \\ &= (-3)^k[(-3)^2 - 9] \\ &= (-3)^k(0) = 0 \quad \text{for all } k \end{aligned}$$

4. The signals 3^k and $(-3)^k$ are linearly independent because neither is a multiple of the other. If H is the solution space of the difference equation in Exercise 2, then $\dim H = 2$, by Theorem 17. So the two linearly independent solutions form a basis for H, by the Basis Theorem in Section 4.5.

6. If $y_k = 5^k \cos\left(\frac{k\pi}{2}\right)$, then

$$\begin{aligned} y_{k+2} + 25y_k &= 5^{k+2}\cos\left(\frac{(k+2)\pi}{2}\right) + 25 \cdot 5^k \cos\left(\frac{k\pi}{2}\right) \\ &= 5^{k+2}\left[\cos\left(\frac{k\pi}{2} + \pi\right) + \cos\left(\frac{k\pi}{2}\right)\right] \\ &= 0 \quad \text{for all } k \end{aligned}$$

because $\cos(t + \pi) = -\cos t$ for all t. A similar calculation holds for $z_k = 5^k \sin\left(\frac{k\pi}{2}\right)$, using the trigonometric identity $\sin(t + \pi) = -\sin t$. Thus y_k and z_k are both solutions of the difference equation $y_{k+2} + 25y_k = 0$. These solutions are obviously linearly independent because neither is a multiple of the other. Since the solution space H is two-dimensional, y_k and z_k form a basis for H, by the Basis Theorem.

8. Yes **10.** Yes

12. No two signals cannot span a four-dimensional solution space.

14. $3^k, 4^k$ **16.** $\left(\frac{1}{4}\right)^k, \left(-\frac{3}{4}\right)^k$

18. The auxiliary equation is $r^2 - 1.35r + .45 = 0$, with roots $.75$ and $.6$. The constant solution of the nonhomogeneous equation is found by solving $T - 1.35T + .45T = 1$, to obtain $T = 10$. The general solution of the nonhomogeneous equation is

$$Y_k = c_1(.75)^k + c_2(.6)^k + 10$$

20. Let $a = -2 + \sqrt{3}$ and $b = -2 - \sqrt{3}$. Then c_1 and c_2 must satisfy

$$\begin{bmatrix} a & b \\ a^N & b^N \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5000 \\ 0 \end{bmatrix}$$

Solving (by row operations or Cramer's rule), we obtain

$$y_k = c_1 a^k + c_2 b^k = \frac{5000}{ab^N - ba^N}(a^k b^N - a^N b^k)$$

22. 1.4, 0, −1.4, −2, −1.4, 0, 1.4, 2, 1.4

This signal is 2 times the signal output by the filter when the input (in Example 3) was $\cos(\pi t/4)$. This is what is to be expected since the filter is *linear*. The output should be 2 times the output from $\cos(\pi t/4)$ plus 1 times the (zero) output from $\cos(3\pi t/4)$.

23. b. **[M]** MATLAB code:

```
pay = 450, y = 10000, m = 0
table = [0 ; y]
while y > 450
     y = 1.01*y - pay
     m = m + 1
     table = [table [m ; y] ]
                 %append new column
end
m, y
```

c. **[M]** At month 26, the last payment is \$114.88. The total paid by the borrower is \$11,364.88.

24. a. $y_{k+1} - 1.005y_k = 200, \quad y_0 = 1{,}000$

b. [M] MATLAB code:

```
pay = 200, y = 1000, table = [0 ; y]
for m = 1:60
     y = 1.005*y + pay
     table = [table  [m ; y] ]
end
interest = y - 60*pay
```

c. [M] The total is \$6213.55 at $k = 24$, \$12,090.06 at $k = 48$, and \$15,302.86 at $k = 60$. When $k = 60$, the interest earned is \$2302.86.

26. $1 + k + c_1 \cdot 5^k + c_2 \cdot 3^k$

28. $2k - 4 + c_1 \cdot (-2)^k + c_2 \cdot 2^{-k}$

30. $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/16 & 0 & 3/4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$$

32. If $a_3 \neq 0$, the order is 3; if $a_3 = 0$ and $a_2 \neq 0$, the order is 2; if $a_3 = a_2 = 0$ and $a_1 \neq 0$, the order is 1; otherwise, the order is 0 (with only the zero signal for a solution).

34. No, the signals could be linearly dependent. *Example:* The following functions are linearly independent when considered as functions on the real line, because they have different periods and no one of the functions is a linear combination of the other two.

$$f(t) = \sin \pi t, \quad g(t) = \sin 2\pi t, \quad h(t) = \sin 3\pi t$$

Since f, g, and h are zero at every integer, the signals are linearly dependent in $\mathbb{S}$.

36. Given $\mathbf{z}$ in V, suppose that $\mathbf{x}_p$ in V satisfies $T(\mathbf{x}_p) = \mathbf{z}$. Also if $\mathbf{u}$ is in the kernel of T, then $T(\mathbf{u}) = \mathbf{0}$. Since T is linear, $T(\mathbf{u} + \mathbf{x}_p) = T(\mathbf{u}) + T(\mathbf{x}_p) = \mathbf{z}$. So the vector $\mathbf{x} = \mathbf{u} + \mathbf{x}_p$ satisfies the nonhomogeneous equation $T(\mathbf{x}) = \mathbf{z}$.

Section 4.9, page 290

2. a.

$$\begin{array}{c} \text{From:} \\ \begin{array}{cc} \text{H} & \text{I} \end{array} \\ \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \end{array} \begin{array}{l} \text{To:} \\ \text{Healthy} \\ \text{Ill} \end{array}$$

b. 15%, 12.5%

c. .925; use $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. a.

$$\begin{array}{c} \text{From:} \\ \begin{array}{ccc} \text{G} & \text{I} & \text{B} \end{array} \\ \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \end{array} \begin{array}{l} \text{To:} \\ \text{Good} \\ \text{Indifferent} \\ \text{Bad} \end{array}$$

b. 20%

c. 48%

6. $\begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$

8. $\begin{bmatrix} 2/5 \\ 1/5 \\ 2/5 \end{bmatrix}$ or $\begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$

10. No, because P^k has a zero in the lower-left corner for all k.

12. a. $\begin{bmatrix} .9 \\ .1 \end{bmatrix}$ **b.** .10, no

14. There is a 50% chance of good weather because $\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$ is the steady-state vector.

16. [M] The steady-state vector is approximately (.435, .091, .474). Of the 2000 cars, about 182 will be rented or available from the downtown location.

18. If $\alpha = \beta = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the steady-state vectors. Otherwise, $\dfrac{1}{\alpha + \beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ is the only steady-state vector.

20. Let $P = [\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n\,]$, so that

$$P^2 = [\,P\mathbf{p}_1 \quad P\mathbf{p}_2 \quad \cdots \quad P\mathbf{p}_n\,]$$

By Exercise 19(c), the columns of P^2 are probability vectors, so P^2 is a stochastic matrix. Alternatively, $SP = S$, by Exercise 19(b), since P is a stochastic matrix. Right-multiplication by P yields $SP^2 = SP$. The right side is just S, so that $SP^2 = S$. Since the entries in P^2 are obviously nonnegative (they are sums of products of the nonnegative entries in P), this shows that P^2 is also a stochastic matrix.

21. [M] a. To four decimal places,

$$P^4 = P^5 = \begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .2255 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$$

Note that, due to round-off, the column sums are not 1.

b. To four decimal places,

$$Q^{80} = \begin{bmatrix} .7354 & .7348 & .7351 \\ .0881 & .0887 & .0884 \\ .1764 & .1766 & .1765 \end{bmatrix},$$

$$Q^{116} = Q^{117} = \begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$$

c. Let P be an $n \times n$ regular stochastic matrix, $\mathbf{q}$ the steady-state vector of P, and $\mathbf{e}_1$ the first column of the identity matrix. Then $P^k\mathbf{e}_1$ is the first column of P^k. By Theorem 18, $P^k\mathbf{e}_1 \to \mathbf{q}$ as $k \to \infty$. Replacing $\mathbf{e}_1$ by the other columns of the identity matrix, we conclude that each column of P^k converges to $\mathbf{q}$ as $k \to \infty$. Thus $P^k \to [\mathbf{q} \quad \mathbf{q} \quad \cdots \quad \mathbf{q}]$.

22. **[M]** (Discussion based on MATLAB Student Version 4.0, running on a 100-MHz 486 laptop computer with 32 Mb of memory) Let A be a random 32×32 stochastic matrix.

Method (1): The following command line will construct A and $\mathbf{q}$ but not display them, and it will announce the elapsed computer processing time and the number of flops used: (type all the commands on one line so they can be recalled and rerun several times)

```
A = randstoc(32); flops(0);
tic, x = nulbasis(A - eye(32));
q = x/sum(x); toc, flops
```

The time ranged from 1.04 to 1.21 seconds, with 35,463 flops.

Method (2):

```
A = randstoc(32); flops(0);
tic, B = A^100; q = B(:,1); toc, flops
```

The time ranged from 1.37 to 1.48 seconds, with 6,488,082 flops. If only A^{70} is computed, the time is about .94 seconds, which is faster than method (1), even though it uses about 4,522,000 flops.

CHAPTER 5

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2. Yes **4.** Yes, $\lambda = 3 + \sqrt{2}$ **6.** Yes, $\lambda = -2$

8. Yes, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

12. $\lambda = 1$: $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$; $\lambda = 5$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

16. $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **18.** 4, 0, -3

20. $\lambda = 0$. Eigenvectors for $\lambda = 0$ have entries that produce linear dependence relations among the columns of A. Any nonzero vector (in $\mathbb{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for example, $(1, 1, -2)$ and $(1, -1, 0)$.

21. **a.** False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.
b. True. See the paragraph before the Invertible Matrix Theorem box.
c. True. See the discussion of equation (3).
d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.
e. False. See the warning after Example 3.

22. **a.** False. The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero*.
b. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the converse of Theorem 2 (for the case $r = 2$).
c. True. See the paragraph after Example 1.
d. False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.
e. True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A - \lambda I$.

24. Any triangular matrix with the same number in both diagonal entries, such as $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$

26. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. However, $A^2\mathbf{x} = \mathbf{0}$ because $A^2 = 0$. Therefore, $\mathbf{0} = \lambda^2\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, we conclude that λ must be zero. Thus each eigenvalue of A is zero.

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

0. By Exercise 29 applied to A^T in place of A, we conclude that s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

2. Suppose T reflects points across a line through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of T. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$.

. You could try to write $\mathbf{x}_0$ as a linear combination of eigenvectors, $\mathbf{v}_1, \dots, \mathbf{v}_p$, of A. If $\lambda_1, \dots, \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, \dots,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p && \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p && \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

[M] $\lambda = -12$: $\begin{bmatrix} 2 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda = 13$: $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}$

[M] $\lambda = 2$: $\begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$;

$\lambda = 3$: $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

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$\lambda^2 - 10\lambda + 16$; 8, 2

$\lambda^2 - 8\lambda + 3$; $4 \pm \sqrt{13}$

$\lambda^2 - 11\lambda + 40$; no real eigenvalues

$\lambda^2 - 10\lambda + 25$; 5

$-\lambda^3 + 14\lambda + 12$

$-\lambda^3 + 5\lambda^2 - 2\lambda - 8$

$-\lambda^3 + 4\lambda^2 + 25\lambda - 28$

5, 1, 1, −4

18. $h = 6$

20. $$\begin{aligned} \det(A^T - \lambda I) &= \det(A^T - \lambda I^T) \\ &= \det(A - \lambda I)^T && \text{Transpose property} \\ &= \det(A - \lambda I) && \text{Theorem 3(c)} \end{aligned}$$

21. **a.** False (although true for a triangular matrix). See Example 1 for a matrix whose determinant is not the product of its diagonal entries.
b. False. However, a row replacement operation does not change the determinant. See Theorem 3.
c. True. See Theorem 3.
d. False. See the solution of Example 4. The monomial $\lambda + 5$ is a factor of the characteristic polynomial if and only if -5 is an eigenvalue of A; it may also happen that 5 is an eigenvalue.

22. **a.** False. The absolute value of $\det A$ equals the volume. See the paragraph before Example 2.
b. False. A and A^T have the same determinant. See Theorem 3.
c. True. See the paragraph before Example 4.
d. False. See the warning after Theorem 4.

24. First observe that if P is invertible, then Theorem 3(b) shows that $1 = \det(I) = \det(PP^{-1}) = (\det P)(\det P^{-1})$. Then, if $A = PBP^{-1}$, Theorem 3(b) again shows that

$$\begin{aligned} \det A &= \det(PBP^{-1}) \\ &= (\det P)(\det B)(\det P^{-1}) = \det B \end{aligned}$$

26. If $a \neq 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & d - ca^{-1}b \end{bmatrix} = U$, and $\det A = (a)(d - ca^{-1}b) = ad - bc$. If $a = 0$, then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = U$ (with one interchange), so $\det A = (-1)^1(cb) = 0 - bc$.

28. **[M]** In general, the eigenvectors of A are not the same as the eigenvectors of A^T, unless, of course, $A^T = A$.

30. **[M]** $a = 32$: $\lambda = 1, 1, 2$;
$a = 31.9$: $\lambda = .2958, 1, 2.7042$;
$a = 31.8$: $\lambda = -.1279, 1, 3.1279$;
$a = 32.1$: $\lambda = 1, 1.5 \pm .9747i$;
$a = 32.2$: $\lambda = 1, 1.5 \pm 1.4663i$

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2. $\dfrac{1}{16}\begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}$

4. $\begin{bmatrix} 4 - 3\cdot 2^k & 12\cdot 2^k - 12 \\ 1 - 2^k & 4\cdot 2^k - 3 \end{bmatrix}$

6. $\lambda = 5$: $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 4$: $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

When an answer involves a diagonalization, $A = PDP^{-1}$, the factors P and D are not unique, so your answer may differ from that given here.

8. Not diagonalizable

10. $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

12. $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

14. $P = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

16. $P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. $P = \begin{bmatrix} -4 & 1 & -2 \\ 3 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

20. $P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

21. a. False. The symbol D does not automatically denote a diagonal matrix.

b. True. See the remark after the statement of the Diagonalization Theorem.

c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.

d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem given in this section.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. a. False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.

b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)

c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.

d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If $\mathbf{v}$ is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

28. If A has n linearly independent eigenvectors, then by Theorem 5, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned} A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} D P^T = QDQ^{-1} \end{aligned}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By Theorem 5, the columns of Q are n linearly independent eigenvectors of A^T.

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

32. [M] $P = \begin{bmatrix} 28 & 1 & -2 & -1 \\ 28 & 1 & 0 & 0 \\ 36 & -2 & 1 & 0 \\ 5 & 1 & 0 & 1 \end{bmatrix}$,

$D = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

34. [M] $P = \begin{bmatrix} 1 & 0 & -1 & 4 & -2 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 3 & 0 & -1 & 2 & 0 \\ 3 & 0 & 1 & 0 & 2 \end{bmatrix}$,

$D = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

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2. $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$

6. a. $2 - t + 3t^2 - t^3 + t^4$

b. For any $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{P}_2$ and any scalar c,

$$\begin{aligned} T(\mathbf{p}+\mathbf{q}) &= [\mathbf{p}(t)+\mathbf{q}(t)] + t^2[\mathbf{p}(t)+\mathbf{q}(t)] \\ &= [\mathbf{p}(t)+t^2\mathbf{p}(t)] + [\mathbf{q}(t)+t^2\mathbf{q}(t)] \\ &= T(\mathbf{p}) + T(\mathbf{q}) \end{aligned}$$

$$\begin{aligned} T(c\mathbf{p}) &= [c\cdot\mathbf{p}(t)] + t^2[c\cdot\mathbf{p}(t)] = c\cdot[\mathbf{p}(t)+t^2\mathbf{p}(t)] \\ &= c\cdot T(\mathbf{p}) \end{aligned}$$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$

0. a. For any $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{P}_3$ and any scalar c,

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-3) \\ (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-3) \\ \mathbf{q}(-1) \\ \mathbf{q}(1) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-3) \\ (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(3) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} = c\cdot T(\mathbf{p})$$

b. $\begin{bmatrix} 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

If there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 3. In this case, A would have three linearly independent eigenvectors. However, this is not necessarily the case, because A has only two distinct eigenvalues.

If $A = PBP^{-1}$, then $A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB\cdot I\cdot BP^{-1} = PB^2P^{-1}$. So A^2 is similar to B^2.

22. If A is diagonalizable, then $A = PDP^{-1}$ for some P. Also, if B is similar to A, then $B = QAQ^{-1}$ for some Q. Then

$$\begin{aligned} B &= Q(PDP^{-1})Q^{-1} = (QP)D(P^{-1}Q^{-1}) \\ &= (QP)D(QP)^{-1} \end{aligned}$$

So B is diagonalizable.

24. If $A = PBP^{-1}$, then rank A = rank $P(BP^{-1})$ = rank BP^{-1}, by Supplementary Exercise 10 in Chapter 4. Also, rank BP^{-1} = rank B, by Supplementary Exercise 11 in Chapter 4, since P^{-1} is invertible. Thus rank A = rank B.

26. If $A = PDP^{-1}$ for some P, then the general trace property from Exercise 25 shows that $\operatorname{tr} A = \operatorname{tr}[(PD)P^{-1}] = \operatorname{tr}[P^{-1}PD] = \operatorname{tr} D$. (Or, one can use the result of Exercise 25 that since A is similar to D, $\operatorname{tr} A = \operatorname{tr} D$.) Since the eigenvalues of A are on the main diagonal of D, $\operatorname{tr} D$ is the sum of the eigenvalues of A.

28. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$, and $[I(\mathbf{b}_j)]_{\mathcal{C}} = [\mathbf{b}_j]_{\mathcal{C}}$. By formula (4), the matrix for I relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$M = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

In Theorem 15 of Section 4.7, this matrix was denoted by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and was called the *change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

30. **[M]** $P^{-1}AP = \begin{bmatrix} 8 & 3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$

32. **[M]** $\lambda = 2$: $\mathbf{b}_1 = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}$; $\lambda = 4$: $\mathbf{b}_2 = \begin{bmatrix} -30 \\ -7 \\ 3 \\ 0 \end{bmatrix}$,

$\mathbf{b}_3 = \begin{bmatrix} 39 \\ 5 \\ 0 \\ 3 \end{bmatrix}$; $\lambda = 5$: $\mathbf{b}_4 = \begin{bmatrix} 11 \\ -3 \\ 4 \\ 4 \end{bmatrix}$;

basis: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$

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2. $\lambda = 3 + i$, $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$; $\lambda = 3 - i$, $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

4. $\lambda = 4 + i$, $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$; $\lambda = 4 - i$, $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

6. $\lambda = 4 - 3i$, $\begin{bmatrix} i \\ 1 \end{bmatrix}$; $\lambda = 4 + 3i$, $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

8. $\lambda = \sqrt{3} \pm 3i$, $\varphi = -\pi/3$ radian, $r = \sqrt{12} = 2\sqrt{3}$

10. $\lambda = -5 \pm 5i$, $\varphi = 3\pi/4$ radians, $r = 5\sqrt{2}$

12. $\lambda = \pm .3i$, $\varphi = -\pi/2$ radians, $r = .3$

In Exercises 13–20, other answers are possible. Any P that makes $P^{-1}AP$ equal to the given C or to C^T is a satisfactory answer. First find P; then compute $P^{-1}AP$.

14. $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

16. $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

18. $P = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

20. $P = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} .28 & -.96 \\ .96 & .28 \end{bmatrix}$

22. $A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = \mu(\lambda\mathbf{x}) = \lambda(\mu\mathbf{x})$

24. $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda \cdot \overline{\mathbf{x}}^T\mathbf{x}$ because $\mathbf{x}$ is an eigenvector. It is easy to see that $\overline{\mathbf{x}}^T\mathbf{x}$ is real (and positive) because $\bar{z}z$ is nonnegative for every complex number z. Since $\overline{\mathbf{x}}^T A\mathbf{x}$ is real, by Exercise 23, so is λ. Next, write $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then

$$A\mathbf{x} = A(\mathbf{u} + i\mathbf{v}) = A\mathbf{u} + iA\mathbf{v} \quad \text{and} \quad \lambda\mathbf{x} = \lambda\mathbf{u} + i\lambda\mathbf{v}$$

The real part of $A\mathbf{x}$ is $A\mathbf{u}$ because the entries in A, $\mathbf{u}$, and $\mathbf{v}$ are all real. The real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because λ and the entries in $\mathbf{u}$ and $\mathbf{v}$ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are equal, too. (Apply the corresponding statement about complex numbers to each entry of $A\mathbf{x}$.) Thus $A\mathbf{u} = \lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

26. a. If $\lambda = a - bi$, then

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &= (a - bi)(\operatorname{Re}\mathbf{v} + i\operatorname{Im}\mathbf{v}) \\ &= \underbrace{(a\operatorname{Re}\mathbf{v} + b\operatorname{Im}\mathbf{v})}_{\operatorname{Re}A\mathbf{v}} + i\underbrace{(a\operatorname{Im}\mathbf{v} - b\operatorname{Re}\mathbf{v})}_{\operatorname{Im}A\mathbf{v}} \end{aligned}$$

By Exercise 25,

$$\begin{aligned} A(\operatorname{Re}\mathbf{v}) &= \operatorname{Re}A\mathbf{v} = a\operatorname{Re}\mathbf{v} + b\operatorname{Im}\mathbf{v} \\ A(\operatorname{Im}\mathbf{v}) &= \operatorname{Im}A\mathbf{v} = -b\operatorname{Re}\mathbf{v} + a\operatorname{Im}\mathbf{v} \end{aligned}$$

b. Let $P = [\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}]$. By (a),

$$A(\operatorname{Re}\mathbf{v}) = P\begin{bmatrix} a \\ b \end{bmatrix}, A(\operatorname{Im}\mathbf{v}) = P\begin{bmatrix} -b \\ a \end{bmatrix}$$

So

$$\begin{aligned} AP &= [A(\operatorname{Re}\mathbf{v}) \quad A(\operatorname{Im}\mathbf{v})] \\ &= \left[P\begin{bmatrix} a \\ b \end{bmatrix} \quad P\begin{bmatrix} -b \\ a \end{bmatrix}\right] = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PC \end{aligned}$$

28. [M] $P = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix},$

$C = \begin{bmatrix} -.4 & -1.0 & 0 & 0 \\ 1.0 & -.4 & 0 & 0 \\ 0 & 0 & -.2 & -.5 \\ 0 & 0 & .5 & -.2 \end{bmatrix}$

Other choices are possible, but C must equal $P^{-1}AP$.

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2. $\mathbf{x}_k = 2 \cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 1 \cdot \left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + 2 \cdot \left(\frac{3}{5}\right)^k \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$

So $\mathbf{x}_k \approx 2 \cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ for all k sufficiently large.

4. If $p = .125$, the eigenvalues are 1 and .6. An eigenvector for $\lambda = 1$ is (4, 5), so eventually the sizes of the two populations will stabilize, and there will be 4 spotted owls for every 5 (thousand) wood rats.

6. If $p = .5$, the eigenvalues are .9 and .7, and

$$\mathbf{x}_k = c_1(.9)^k \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_2(.7)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \mathbf{0} \quad \text{as } k \to \infty$$

So both populations perish. If $p = .4$, the eigenvalues are 1 and .6, and

$$\mathbf{x}_k = c_1(1)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2(.6)^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} \to c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{as } k \to \infty$$

Eventually, for every spotted owl, there will be about 2000 flying squirrels.

8. Saddle point (because one or more eigenvalues are greater than 1, and one or more eigenvalues are less than 1, in magnitude); direction of greatest repulsion: the line through (0, 0, 0) and (1, 0, −3); direction of greatest attraction: the line through (0, 0, 0) and (−3, −3, 7)

10. Attractor; eigenvalues: .9, .5; direction of greatest attraction: the line through (0, 0) and (2, 1)

12. Saddle point; eigenvalues: 1.1, .8; greatest repulsion: line through (0, 0) and (1, 1); greatest attraction: line through (0, 0) and (2, 1)

14. Repellor; eigenvalues: 1.3, 1.1; greatest repulsion: line through (0, 0) and (−3, 2)

6. **[M]** $\mathbf{v}_k = c_1\begin{bmatrix}.435\\.091\\.474\end{bmatrix} + c_2(.89)^k\begin{bmatrix}-1\\1\\0\end{bmatrix} + c_3(.81)^k\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Note: The exact value of the steady-state vector is $\mathbf{q} = (91/209, 19/209, 99/209) \approx (.435, .091, .474)$.

8. **a.** $A = \begin{bmatrix}0 & 0 & .42\\.6 & 0 & 0\\0 & .75 & .95\end{bmatrix}$

b. [M] The long-term growth rate is $\lambda_1 = 1.105$. A corresponding eigenvector is approximately (38, 21, 100). For each 100 adults, there will be approximately 38 calves and 21 yearlings.

ection 5.7, page 355

2. $\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t} + \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$

. $\frac{13}{4}\begin{bmatrix}-1\\1\end{bmatrix}e^{3t} - \frac{5}{4}\begin{bmatrix}-5\\1\end{bmatrix}e^{-t}$. The origin is a saddle point. The direction of greatest attraction is the line through $(-5, 1)$ and the origin. The direction of greatest repulsion is the line through $(-1, 1)$ and the origin.

. $-\begin{bmatrix}2\\3\end{bmatrix}e^{-2t} + 5\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$. The origin is an attractor. The direction of greatest attraction is the line through (2, 3) and the origin.

. Set $P = \begin{bmatrix}2 & 1\\3 & 1\end{bmatrix}$ and $D = \begin{bmatrix}-2 & 0\\0 & -1\end{bmatrix}$. Then $A = PDP^{-1}$. Substituting $\mathbf{x} = P\mathbf{y}$ into $\mathbf{x}' = A\mathbf{x}$, we have

$$\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y})$$
$$P\mathbf{y}' = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$$

Left-multiplying by P^{-1} gives

$\mathbf{y}' = D\mathbf{y}$, or $\begin{bmatrix}y_1'(t)\\y_2'(t)\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}$

(complex): $c_1\begin{bmatrix}1+i\\-2\end{bmatrix}e^{(2+i)t} + c_2\begin{bmatrix}1-i\\-2\end{bmatrix}e^{(2-i)t}$

(real): $c_1\begin{bmatrix}\cos t - \sin t\\-2\cos t\end{bmatrix}e^{2t} + c_2\begin{bmatrix}\sin t + \cos t\\-2\sin t\end{bmatrix}e^{2t}$

The trajectories spiral out, away from the origin.

(complex): $c_1\begin{bmatrix}3-i\\2\end{bmatrix}e^{(-1+2i)t} + c_2\begin{bmatrix}3+i\\2\end{bmatrix}e^{(-1-2i)t}$

(real):

$c_1\begin{bmatrix}3\cos 2t + \sin 2t\\2\cos 2t\end{bmatrix}e^{-t} + c_2\begin{bmatrix}3\sin 2t - \cos 2t\\2\sin 2t\end{bmatrix}e^{-t}$

The trajectories spiral in, toward the origin.

14. (complex): $c_1\begin{bmatrix}1-i\\4\end{bmatrix}e^{2it} + c_2\begin{bmatrix}1+i\\4\end{bmatrix}e^{-2it}$

(real): $c_1\begin{bmatrix}\cos 2t + \sin 2t\\4\cos 2t\end{bmatrix} + c_2\begin{bmatrix}\sin 2t - \cos 2t\\4\sin 2t\end{bmatrix}$

The trajectories are ellipses about the origin.

16. **[M]** $\mathbf{x}(t) = c_1\begin{bmatrix}7\\-2\\3\end{bmatrix}e^{4t} + c_2\begin{bmatrix}3\\-1\\1\end{bmatrix}e^{3t} + c_3\begin{bmatrix}2\\0\\1\end{bmatrix}e^{2t}$

The origin is a repellor. All trajectories curve away from the origin.

18. **[M]** (complex):

$c_1\begin{bmatrix}1\\2\\0\end{bmatrix}e^{-7t} + c_2\begin{bmatrix}6+2i\\9+3i\\10\end{bmatrix}e^{(5+i)t} + c_3\begin{bmatrix}6-2i\\9-3i\\10\end{bmatrix}e^{(5-i)t}$

(real): $c_1\begin{bmatrix}1\\2\\0\end{bmatrix}e^{-7t} + c_2\begin{bmatrix}6\cos t - 2\sin t\\9\cos t - 3\sin t\\10\cos t\end{bmatrix}e^{5t} +$

$c_3\begin{bmatrix}6\sin t + 2\cos t\\9\sin t + 3\cos t\\10\sin t\end{bmatrix}e^{5t}$

When $c_2 = c_3 = 0$, the trajectories tend straight toward **0**. In other cases, the trajectories spiral outward.

20. **[M]** $A = \begin{bmatrix}-2 & 1/3\\3/2 & -3/2\end{bmatrix}$,

$\begin{bmatrix}v_1(t)\\v_2(t)\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\3\end{bmatrix}e^{-t} - \frac{2}{3}\begin{bmatrix}-2\\3\end{bmatrix}e^{-2.5t}$

22. **[M]** $A = \begin{bmatrix}0 & .2\\-.4 & -.8\end{bmatrix}$,

$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix} = \begin{bmatrix}30\sin .8t\\12\cos .8t - 6\sin .8t\end{bmatrix}e^{-.4t}$

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2. Eigenvector: $\mathbf{x}_4 = \begin{bmatrix}-.2520\\1\end{bmatrix}$, or $A\mathbf{x}_4 = \begin{bmatrix}-1.2536\\5.0064\end{bmatrix}$; $\lambda \approx 5.0064$

4. Eigenvector: $\mathbf{x}_4 = \begin{bmatrix}1\\.7502\end{bmatrix}$, or $A\mathbf{x}_4 = \begin{bmatrix}-.4012\\-.3009\end{bmatrix}$; $\lambda \approx -.4012$

6. $\mathbf{x} = \begin{bmatrix}-.4996\\1\end{bmatrix}$, $A\mathbf{x} = \begin{bmatrix}-2.0008\\4.0024\end{bmatrix}$; estimated $\lambda = 4.0024$

8. **[M]** $\mathbf{x}_k$: $\begin{bmatrix}.5\\1\end{bmatrix}, \begin{bmatrix}.2857\\1\end{bmatrix}, \begin{bmatrix}.2558\\1\end{bmatrix}, \begin{bmatrix}.2510\\1\end{bmatrix}, \begin{bmatrix}.2502\\1\end{bmatrix}$

μ_k: 7, 6.14, 6.02, 6.0039, 6.0006

10. **[M]** $\mu_5 = 9.9319$, $\mu_6 = 9.9872$; actual value: 10
Note: Starting with $\mathbf{x}_0 = (0, 0, 1)$ produces $\mu_4 = 9.9993$, $\mu_5 = 9.9999$.

12. μ_k: $-4.3333, -3.9231, -4.0196, -3.9951$
$R(\mathbf{x}_k)$: $-3.9231, -3.9951, -3.9997, -3.99998$

14. Use the inverse power method, with $\alpha = 4$.

16. $\lambda = \alpha + 1/\mu$

18. **[M]** $\nu_0 = -1.375$, $\nu_1 = -1.42623$, $\nu_2 = -1.42432$, $\nu_3 = -1.42444$. Actual: -1.424429 (accurate to 6 places)

20. **[M]** **a.** $\mu_8 = 19.1820 = \mu_9$ to four decimal places. To six places, the largest eigenvalue is 19.182037, with eigenvector (.184416, 1, .179615, .407110).

b. $\mu_1^{-1} = .012235$, $\mu_2^{-1} = .012205$. To six places, the smallest eigenvalue is .012206, with eigenvector (1, .222610, −.917993, .660483). The other eigenvalues are −2.453128 and −1.741114, to six places.

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20. **[M]** The MATLAB command `[P D] = eig(A)` produces a matrix P whose condition number is 1.6×10^8, and a diagonal matrix D whose diagonal entries are almost 2, 2, 1. (The exact eigenvalues of A are 2, 2, 1.)

21. **[M]** $A^4 = 0$, so the only eigenvalue is $\lambda = 0$. However, MATLAB reports two small complex eigenvalues, each with multiplicity 2.

CHAPTER 6

Section 6.1, page 376

2. 35, 5, $\frac{1}{7}$ **4.** $\begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$ **6.** $\begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

8. 7 **10.** $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$ **12.** $\begin{bmatrix} .8 \\ .6 \end{bmatrix}$

14. $2\sqrt{17}$ **16.** Orthogonal **18.** Not orthogonal

19. **a.** True. See the definition of $\|\mathbf{v}\|$.
b. True. See Theorem 1(c).
c. True. See the discussion of Fig. 5.
d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
e. True. See the box following Example 6.

20. **a.** True. See Example 1 and Theorem 1(a).
b. False. The absolute value is missing. See the box before Example 2.
c. True, by definition of the orthogonal complement.
d. True, by the Pythagorean Theorem.
e. True, by Theorem 3.

22. $\mathbf{u}\cdot\mathbf{u} \geq 0$ because $\mathbf{u}\cdot\mathbf{u}$ is a sum of squares of the entries in $\mathbf{u}$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

24.
$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\ \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot(-\mathbf{v}) - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \end{aligned}$$
When $\|\mathbf{u}+\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2$ are added, the $\mathbf{u}\cdot\mathbf{v}$ terms cancel, and the result is $2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

26. Theorem 2 in Chapter 4, because W is the null space of the $1 \times n$ matrix $\mathbf{u}^T$. W is a plane through the origin.

28. An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y} = 0$ and $\mathbf{v}\cdot\mathbf{y} = 0$. By linearity of the inner product [Theorem 1(b) and 1(c)],
$$\mathbf{w}\cdot\mathbf{y} = (c_1\mathbf{u} + c_2\mathbf{v})\cdot\mathbf{y} = c_1\mathbf{u}\cdot\mathbf{y} + c_2\mathbf{v}\cdot\mathbf{y} = c_1 0 + c_2 0 = 0$$

30. **a.** If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u} = c(\mathbf{z}\cdot\mathbf{u}) = c0 = 0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.
b. Take any $\mathbf{z}_1, \mathbf{z}_2$ in $W^\perp$. Then, for any $\mathbf{u}$ in W, $(\mathbf{z}_1 + \mathbf{z}_2)\cdot\mathbf{u} = \mathbf{z}_1\cdot\mathbf{u} + \mathbf{z}_2\cdot\mathbf{u} = 0 + 0 = 0$, which shows that $\mathbf{z}_1 + \mathbf{z}_2$ is in $W^\perp$.
c. Obviously $\mathbf{0}$ is in $W^\perp$, because $\mathbf{0}$ is orthogonal to every vector. This fact, together with (a) and (b), shows that $W^\perp$ is a subspace.

32. **[M]** This exercise anticipates Theorem 7 in Section 6.2. The matrix A has orthonormal columns.

33. **[M]** The mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\dfrac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ is a linear transformation. In Section 6.2, the mapping will be called the orthogonal projection of $\mathbf{x}$ onto Span $\{\mathbf{v}\}$. To verify the linearity, take any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^4$(or $\mathbb{R}^n$) and any scalar c. Then properties of the inner product (Theorem 1) show that
$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= \frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{v} + \mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \\ &= \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$
$$T(c\mathbf{x}) = \frac{(c\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{c(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = c\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = cT(\mathbf{x})$$

Another argument is to view T as the composition of three linear mappings: $\mathbf{x} \mapsto a = \mathbf{x}\cdot\mathbf{v}$, $a \mapsto b = a/(\mathbf{v}\cdot\mathbf{v})$, and $b \mapsto b\mathbf{v}$.

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2. Orthogonal

4. Orthogonal

6. Not orthogonal

8. Show $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, mention Theorem 4, and observe that two linearly independent vectors in $\mathbb{R}^2$ form a basis. Then obtain

$$\mathbf{x} = -\tfrac{15}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \tfrac{30}{40}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = -\tfrac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \tfrac{3}{4}\begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

10. Show $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$. Mention Theorem 4, and observe that three linearly independent vectors in $\mathbb{R}^3$ form a basis. Then obtain

$$\mathbf{x} = \tfrac{24}{18}\mathbf{u}_1 + \tfrac{3}{9}\mathbf{u}_2 + \tfrac{6}{18}\mathbf{u}_3 = \tfrac{4}{3}\mathbf{u}_1 + \tfrac{1}{3}\mathbf{u}_2 + \tfrac{1}{3}\mathbf{u}_3$$

12. $\begin{bmatrix} .4 \\ -1.2 \end{bmatrix}$

14. $\mathbf{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

16. $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, distance is $\sqrt{45} = 3\sqrt{5}$

18. Not orthogonal

20. $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

22. Orthonormal

23. **a.** True. For example, the vectors $\mathbf{u}$ and $\mathbf{y}$ in Example 3 are linearly independent but not orthogonal.
b. True. The formulas for the weights are given in Theorem 5.
c. False. See the paragraph following Example 5.
d. False. The matrix must also be square. See the paragraph before Example 7.
e. False. See Example 4. The distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$.

24. **a.** True. But every orthogonal set of *nonzero vectors* is linearly independent. See Theorem 4.
b. False. To be orthonormal, the vectors in S must be unit vectors as well as being orthogonal to each other.
c. True. See Theorem 7(a).
d. True. See the paragraph before Example 3.
e. True. See the paragraph before Example 7.

26. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are nonzero and orthogonal, then they are linearly independent, by Theorem 4. By the Invertible Matrix Theorem, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$. If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then W must be $\mathbb{R}^n$.

28. If U is an $n \times n$ orthogonal matrix, then $I = UU^{-1} = UU^T$. Since U is the transpose of U^T, Theorem 6 applied to U^T says that U^T has orthonormal columns. In particular, the columns of U^T are linearly independent and hence form a basis for $\mathbb{R}^n$, by the Invertible Matrix Theorem (see Section 4.6). That is, the rows of U form a basis (in fact, an orthonormal basis) for $\mathbb{R}^n$.

30. If U is an orthogonal matrix, its columns are orthonormal. Interchanging the columns does not change their orthonormality, so the new matrix—say, V—still has orthonormal columns. By Theorem 6, $V^TV = I$. Since V is square, $V^T = V^{-1}$ by the Invertible Matrix Theorem.

32. If $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, then by Theorem 1(c) in Section 6.1,
$(c_1\mathbf{v}_1)\cdot(c_2\mathbf{v}_2) = c_1[\mathbf{v}_1\cdot(c_2\mathbf{v}_2)] = c_1c_2(\mathbf{v}_1\cdot\mathbf{v}_2) = c_1c_20 = 0$.

33. **[M]** The proof of Theorem 6 shows that the inner products to be checked are actually entries in the matrix product A^TA. A calculation shows that $A^TA = 100I_4$. Since the off-diagonal entries in A^TA are zero, the columns of A are orthogonal.

34. **[M]** **a.** $U^TU = I_4$, but UU^T is an 8×8 matrix which is nothing like I_8. In fact

$$UU^T = (.01)\begin{bmatrix} 82 & 0 & -20 & 8 & 6 & 20 & 24 & 0 \\ 0 & 42 & 24 & 0 & -20 & 6 & 20 & -32 \\ -20 & 24 & 58 & 20 & 0 & 32 & 0 & 6 \\ 8 & 0 & 20 & 82 & 24 & -20 & 6 & 0 \\ 6 & -20 & 0 & 24 & 18 & 0 & -8 & 20 \\ 20 & 6 & 32 & -20 & 0 & 58 & 0 & 24 \\ 24 & 20 & 0 & 6 & -8 & 0 & 18 & -20 \\ 0 & -32 & 6 & 0 & 20 & 24 & -20 & 42 \end{bmatrix}$$

b. The vector $\mathbf{p} = UU^T\mathbf{y}$ is in Col U because $\mathbf{p} = U(U^T\mathbf{y})$. Since the columns of U are simply scaled versions of the columns of A, Col U = Col A. Thus $\mathbf{p}$ is in Col A.
d. From (c), $\mathbf{z}$ is orthogonal to each column of A. By Exercise 29 in Section 6.1, $\mathbf{z}$ must be orthogonal to every vector in Col A; that is, $\mathbf{z}$ is in $(\text{Col}\,A)^\perp$.

Section 6.3, page 395

2. $\mathbf{v} = 2\mathbf{u}_1 + \tfrac{3}{7}\mathbf{u}_2 + \tfrac{12}{7}\mathbf{u}_3 - \tfrac{8}{7}\mathbf{u}_4$; $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ \vdots \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \\ \vdots \end{bmatrix}$

4. $\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = \mathbf{y}$ **8.** $\mathbf{y} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$

10. $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$ **16.** 8

18. a. $U^TU = [1] = 1$, $UU^T = \begin{bmatrix} .1 & -.3 \\ -.3 & .9 \end{bmatrix}$

b. $\text{proj}_W \mathbf{y} = \dfrac{-20}{\sqrt{10}}\mathbf{u}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$,

$(UU^T)\mathbf{y} = \begin{bmatrix} .7 - 2.7 \\ -2.1 + 8.1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

20. Any multiple of $\begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}$, such as $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

21. a. True. See the calculations for $\mathbf{z}_2$ in Example 1 or the box after Example 6 in Section 6.1.

b. True, by the Orthogonal Decomposition Theorem.

c. False. See the last paragraph in the proof of Theorem 8, or see the second paragraph after the statement of Theorem 9.

d. True. See the box before The Best Approximation Theorem.

e. True. Theorem 10 applies to the column space W of U because the columns of U are linearly independent and hence form a basis for W.

22. a. True. See the proof of the Orthogonal Decomposition Theorem.

b. True. See the subsection "A Geometric Interpretation of the Orthogonal Projection."

c. True, by the uniqueness of the orthogonal decomposition in Theorem 8.

d. False. The Best Approximation Theorem says that the best approximation to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$.

e. False, unless $n = p$, because $UU^T\mathbf{x}$ is only the orthogonal projection of $\mathbf{x}$ onto the column space of U. See the paragraph following Theorem 10.

24. a. By hypothesis, the vectors $\mathbf{w}_1, \dots, \mathbf{w}_p$ are pairwise orthogonal, and the vectors $\mathbf{v}_1, \dots, \mathbf{v}_q$ pairwise orthogonal. Also, $\mathbf{w}_i \cdot \mathbf{v}_j = 0$ for any i and j because the orthogonal ... complement of W.

b. For any $\mathbf{y}$ in $\mathbb{R}^n$, write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ as in the Orthogonal Decomposition Theorem, with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. Then there exist scalars $c_1, \dots, c_p$ and $d_1, \dots, d_q$ such that

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c_1\mathbf{w}_1 + \cdots + c_p\mathbf{w}_p + d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q$$

Thus $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ spans $\mathbb{R}^n$.

c. The set $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is linearly independent by (a), spans $\mathbb{R}^n$ by (b), and thus is a basis for $\mathbb{R}^n$. Hence

$$\dim W + \dim W^\perp = p + q = \dim \mathbb{R}^n = n$$

25. [M] U has orthonormal columns, by Theorem 6 in Section 6.2, because $U^TU = I_4$. The closest point to $\mathbf{y}$ in Col U is the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Col U. From Theorem 10,

$$\hat{\mathbf{y}} = UU^T\mathbf{y} = (1.2, .4, 1.2, 1.2, .4, 1.2, .4, .4)$$

26. [M] To two decimal places, $\hat{\mathbf{b}} = UU^T\mathbf{b} = (.20, .92, .44, 1.00, -.20, -.44, .60, -.92)$. The distance from $\mathbf{b}$ to Col U is $\|\mathbf{b} - \hat{\mathbf{b}}\| = 2.1166$, to four decimal places.

Section 6.4, page 402

2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

10. $\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

14. $R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$

16. $Q = \begin{bmatrix} 1/2 & -1/\sqrt{8} & 1/2 \\ -1/2 & 1/\sqrt{8} & 1/2 \\ 0 & 2/\sqrt{8} & 0 \\ 1/2 & 1/\sqrt{8} & -1/2 \\ 1/2 & 1/\sqrt{8} & 1/2 \end{bmatrix}$,

$$R = \begin{bmatrix} 2 & 8 & 7 \\ 0 & \sqrt{8} & 12/\sqrt{8} \\ 0 & 0 & 6 \end{bmatrix}$$

17. a. False. Scaling was used in Example 2, but the scale factor was nonzero.
b. True. See (2) in the statement of Theorem 11.
c. True. See the solution of Example 4.

18. a. False. The three orthogonal vectors must be *nonzero* to be a basis for a three-dimensional subspace. (This was the case in Step 3 of the solution of Example 2.)
b. True. If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x}$ cannot equal $\text{proj}_W\,\mathbf{x}$, because $\text{proj}_W\,\mathbf{x}$ is in W. This idea was used for $\mathbf{x} = \mathbf{v}_{k+1}$ in the proof of Theorem 11.
c. True, by Theorem 12.

20. If $\mathbf{y}$ is in Col A, then $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$. Then $\mathbf{y} = QR\mathbf{x} = Q(R\mathbf{x})$, which shows that $\mathbf{y}$ is a linear combination of the columns of Q using the entries in $R\mathbf{x}$ as weights. Conversely, suppose $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Since R is invertible, the equation $A = QR$ implies that $Q = AR^{-1}$. So $\mathbf{y} = AR^{-1}\mathbf{x} = A(R^{-1}\mathbf{x})$, which shows that $\mathbf{y}$ is in Col A.

22. [M] $\begin{bmatrix} -10 \\ 2 \\ -6 \\ 16 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix}$

23. [M] $Q = \begin{bmatrix} -.5 & .5 & .5774 & 0 \\ .1 & .5 & 0 & .7071 \\ -.3 & -.5 & .5774 & 0 \\ .8 & 0 & .5774 & 0 \\ .1 & .5 & 0 & -.7071 \end{bmatrix}$

$$R = \begin{bmatrix} 20 & -20 & -10 & 10 \\ 0 & 6 & -8 & -6 \\ 0 & 0 & 10.3923 & -5.1962 \\ 0 & 0 & 0 & 7.0711 \end{bmatrix}$$

24. [M] In MATLAB, when A has n columns, suitable commands are

```
Q = A(:,1)/norm(A(:,1))
      % The first column of Q
      for j = 2:n
         v = A(:,j) - Q*(Q'*A(:,j))
         Q(:,j) = v/norm(v)
         % Add a new column to Q
      end
```

Section 6.5, page 411

2. a. $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

4. a. $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. $\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ **8.** $\sqrt{6}$

10. a. $\hat{\mathbf{b}} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$

12. a. $\begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$

14. $A\mathbf{u} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}, A\mathbf{v} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}, \mathbf{b} - A\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix},$

$\mathbf{b} - A\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$. Note that

$\|\mathbf{b} - A\mathbf{u}\| = \|\mathbf{b} - A\mathbf{v}\| = \sqrt{24}$, so $A\mathbf{u}$ and $A\mathbf{v}$ are equally close to $\mathbf{b}$. The orthogonal projection is the *unique* closest point in Col A to $\mathbf{b}$, so neither $A\mathbf{u}$ nor $A\mathbf{v}$ can be $\hat{\mathbf{b}}$. That is, neither $\mathbf{u}$ nor $\mathbf{v}$ can be a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

16. $\hat{\mathbf{x}} = \begin{bmatrix} 2.9 \\ .9 \end{bmatrix}$

17. a. True. See the beginning of the section. The distance from $A\mathbf{x}$ to $\mathbf{b}$ is $\|A\mathbf{x} - \mathbf{b}\|$.
b. True. See the comments about equation (1).
c. False. The inequality points in the wrong direction. See the definition of a least-squares solution.
d. True. See Theorem 13.
e. True. See Theorem 14.

18. a. True. See the paragraph following the definition of a least-squares solution.
b. False. If $\hat{\mathbf{x}}$ is the least-squares solution, then $A\hat{\mathbf{x}}$ is the point in the column space of A closest to $\mathbf{b}$. See Fig. 1 and the paragraph preceding it.
c. True. See the discussion following equation (1).
d. False. The formula applies only when the columns of A are linearly independent. See Theorem 14.
e. False. See the comments after Example 4.
f. False. See the Numerical Note on p. 410.

20. Suppose that $A\mathbf{x} = \mathbf{0}$. Then $A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. Since A^TA is invertible, by hypothesis, $\mathbf{x}$ must be zero. Hence the columns of A are linearly independent.

22. A^TA has n columns because A does. Then

$\text{rank}\, A^TA$	$= n - \dim \text{Nul}\, A^TA$	The Rank Theorem
	$= n - \dim \text{Nul}\, A$	Exercise 19
	$= \text{rank}\, A$	The Rank Theorem

24. $\hat{\mathbf{x}} = A^T\mathbf{b}$, from the normal equations, because $A^TA = I$.

26. **[M]** $a_0 = a_2 = .35\underline{35}$, $a_1 = .5$
(With .707 in place of .7, $a_0 = a_2 \approx .35355339$, $a_1 = .5$.)

Section 6.6, page 420

2. $y = -.6 + .7x$

4. $y = 4.3 - .7x$

6. If the columns of X were linearly dependent, then the same dependence relation would hold for the vectors in $\mathbb{R}^3$ formed from the top three entries of the column. In this case, the *Vandermonde* matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

would be noninvertible. However, it can be shown that since x_1, x_2, and x_3 are distinct, this matrix *is* invertible, which means that the columns of X are, in fact, linearly independent. As in Exercise 5, Theorem 14 implies that there is only one least-squares solution of $\mathbf{y} = X\boldsymbol{\beta}$.

One way to show that the 3×3 matrix above is invertible is to show that its determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. Another way is to appeal to Supplementary Exercise 11(b) in Chapter 2.

8. a. $X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}$, $\quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

b. **[M]** $y = .5132x - .03348x^2 + .001016x^3$, using four significant figures in the coefficients. *Note:* If you use .001 as the coefficient of x^3, your graph will fall somewhat below the last three or four data points.

10. a. $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{y} = \begin{bmatrix} 21.34 \\ 20.68 \\ 20.05 \\ 18.87 \\ 18.30 \end{bmatrix}$,

$$X = \begin{bmatrix} e^{-.02(10)} & e^{-.07(10)} \\ e^{-.02(11)} & e^{-.07(11)} \\ e^{-.02(12)} & e^{-.07(12)} \\ e^{-.02(14)} & e^{-.07(14)} \\ e^{-.02(15)} & e^{-.07(15)} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} M_A \\ M_B \end{bmatrix}, \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

b. **[M]** $y = 19.94e^{-.02t} + 10.10e^{-.07t}$, $M_A = 19.94$, $M_B = 10.10$

12. **[M]** $p = 18.55 + 19.23 \ln w$. When w is 100, p is approximately 107.

14. Write the design matrix as $X = [\,\mathbf{1} \quad \mathbf{x}\,]$. Since the residual vector, $\boldsymbol{\epsilon} = \mathbf{y} - X\hat{\boldsymbol{\beta}}$, is orthogonal to $\text{Col}\, X$, we have (using the notation shown just after Exercise 14)

$$\begin{aligned} 0 &= \mathbf{1}\cdot\boldsymbol{\epsilon} = \mathbf{1}\cdot(\mathbf{y} - X\hat{\boldsymbol{\beta}}) = \mathbf{1}^T\mathbf{y} - (\mathbf{1}^TX)\hat{\boldsymbol{\beta}} \\ &= (y_1 + \cdots + y_n) - [\,n \quad \Sigma x\,] \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \\ &= \Sigma y - n\hat{\beta}_0 - \hat{\beta}_1 \Sigma x \end{aligned}$$

Divide by $-n$, move the first term to the left side of the equation, and obtain $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x}$.

16. The determinant of the coefficient matrix of the equations in (7) is $n\Sigma x^2 - (\Sigma x)^2$. Using the 2×2 formula for the inverse of the coefficient matrix, we have

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \frac{1}{n\Sigma x^2 (\Sigma x)^2} \begin{bmatrix} \Sigma x^2 & -\Sigma x \\ -\Sigma x & n \end{bmatrix} \begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$$

Hence

$$\hat{\beta}_0 = \frac{(\Sigma x^2)(\Sigma y) - (\Sigma x)(\Sigma xy)}{n\Sigma x^2 - (\Sigma x)^2}, \quad \hat{\beta}_1 = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\Sigma x^2 - (\Sigma x)^2}$$

Note: A simple algebraic calculation shows that $\Sigma y - (\Sigma x)\hat{\beta}_1 = n\hat{\beta}_0$, which provides a simple formula for $\hat{\beta}_0$, once $\hat{\beta}_1$ is known.

18. $X^TX = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \Sigma x \\ \Sigma x & (\Sigma x)^2 \end{bmatrix}$

This matrix is a diagonal matrix when $\Sigma x = 0$.

20. $\|X\hat{\boldsymbol{\beta}}\|^2 = (X\hat{\boldsymbol{\beta}})^T(X\hat{\boldsymbol{\beta}}) = \hat{\boldsymbol{\beta}}^TX^TX\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^TX^T\mathbf{y}$, because $\hat{\boldsymbol{\beta}}^T$ satisfies the normal equations: $X^TX\hat{\boldsymbol{\beta}} = X^T\mathbf{y}$. Since $\|X\hat{\boldsymbol{\beta}}\|^2 = SS(R)$ and $\mathbf{y}^T\mathbf{y} = \|\mathbf{y}\|^2 = SS(T)$, Exercise 19 shows that

$$SS(E) = SS(T) - SS(R) = \mathbf{y}^T\mathbf{y} - \hat{\boldsymbol{\beta}}^TX^T\mathbf{y}$$

Section 6.7, page 430

2. $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = 4(3)(3) + 5(-2)(-2) = 56$
$\|\mathbf{y}\|^2 = \langle \mathbf{y}, \mathbf{y} \rangle = 4(-2)(-2) + 5(1)(1) = 21$

$\|\mathbf{x}\|^2\|\mathbf{y}\|^2 = 56(21) = 1176$
$\langle \mathbf{x}, \mathbf{y}\rangle = 4(3)(-2) + 5(-2)(1) = -34$
$|\langle \mathbf{x}, \mathbf{y}\rangle|^2 = 1156 < 1176 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2$

4. Polynomials: $3t - t^2 \qquad 3 + 2t^2$

$$\text{Values:} \quad \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$$

$\langle p, q\rangle = -20 + 0 + 10 = -10$

6. $\|p\| = 2\sqrt{5}$, $\|q\| = \sqrt{59}$

8. $\dfrac{\langle q, p\rangle}{\langle p, p\rangle}p(t) = -\frac{10}{20}p(t) = -\frac{1}{2}(3t - t^2) = -\frac{3}{2}t + \frac{1}{2}t^2$

10. Polynomials: $p_0 \qquad p_1 \qquad q \qquad p(t) = t^3$

$$\text{Values:} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -27 \\ -1 \\ 1 \\ 27 \end{bmatrix}$$

$\hat{p}(t) = \frac{0}{4}p_0 + \frac{164}{20}p_1 + \frac{0}{4}q = \frac{41}{5}t$

12. Use Exercise 11 to get $t^3 - \frac{17}{5}t$. Then $5t^3 - 17t$ is also orthogonal to p_0, p_1, p_2, but its vector of values is $(-6, 12, 0, -12, 6)$. Answer: $p_3(t) = \frac{1}{6}(5t^3 - 17t)$.

14. **1.** $\langle \mathbf{u}, \mathbf{v}\rangle = T(\mathbf{u})\cdot T(\mathbf{v})$ — Definition
$= T(\mathbf{v})\cdot T(\mathbf{u})$ — Property of dot product
$= \langle \mathbf{v}, \mathbf{u}\rangle$ — Definition

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = T(\mathbf{u} + \mathbf{v})\cdot T(\mathbf{w})$ — Definition
$= [T(\mathbf{u}) + T(\mathbf{v})]\cdot T(\mathbf{w})$ — Linearity of T
$= T(\mathbf{u})\cdot T(\mathbf{w}) + T(\mathbf{v})\cdot T(\mathbf{w})$ — Property of $\cdot$
$= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$ — Definition

3. $\langle c\mathbf{u}, \mathbf{v}\rangle = T(c\mathbf{u})\cdot T(\mathbf{v})$ — Definition
$= cT(\mathbf{u})\cdot T(\mathbf{v})$ — Linearity of T
$= c\langle \mathbf{u}, \mathbf{v}\rangle$ — Definition

4. $\langle \mathbf{u}, \mathbf{u}\rangle = T(\mathbf{u})\cdot T(\mathbf{u}) \geq 0$ — Property of dot product
If $\mathbf{u} = \mathbf{0}$, then $T(\mathbf{u}) = \mathbf{0}$, because T is linear, and $\langle \mathbf{u}, \mathbf{u}\rangle = 0$. Conversely, if $\langle \mathbf{u}, \mathbf{u}\rangle = 0$, then $T(\mathbf{u})\cdot T(\mathbf{u}) = 0$, and hence $T(\mathbf{u}) = \mathbf{0}$ by a property of the dot product. Since T is one-to-one, $\mathbf{u} = \mathbf{0}$.

16. $\|\mathbf{u} - \mathbf{v}\|^2$
$= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle$
$= \langle \mathbf{u}, \mathbf{u} - \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle$ — Axioms 2 and 3
$= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$ — Axioms 1–3
$= \langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$ — Axiom 1
$= \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$

If $\{\mathbf{u}, \mathbf{v}\}$ is orthonormal, then $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = 1$ and $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. So $\|\mathbf{u} - \mathbf{v}\|^2 = 2$.

18. The calculation in Exercise 16 shows that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$$

Similarly,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2$$

Adding gives $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

20. If $\mathbf{u} = (a, b)$ and $\mathbf{v} = (1, 1)$, then $\|\mathbf{u}\|^2 = a^2 + b^2$, $\|\mathbf{v}\|^2 = 2$, and $|\langle \mathbf{u}, \mathbf{v}\rangle| = |a + b|$. The desired inequality follows when the Cauchy–Schwarz inequality is rewritten as

$$\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{2}\right)^2 \leq \frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}{4}$$

22. $\displaystyle\int_0^1 (5t - 3)(t^3 - t^2)\,dt = \int_0^1 (5t^4 - 8t^3 + 3t^2)\,dt = 0$

24. $\displaystyle\int_0^1 (t^3 - t^2)^2\,dt = \int_0^1 (t^6 - 2t^5 + t^4)\,dt = 1/105$,
$\|g\| = 1/\sqrt{105}$

26. $1, t, 3t^2 - 4$

27. **[M]** $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = -2 + t^2$, $p_3(t) = (-17t + 5t^3)/6$, $p_4(t) = (72 - 155t^2 + 35t^4)/12$

The columns of the following matrix list the values of the respective polynomials at $-2, -1, 0, 1$, and 2:

$$A = \begin{bmatrix} 1 & -2 & 2 & -1 & 1 \\ 1 & -1 & -1 & 2 & -4 \\ 1 & 0 & -2 & 0 & 6 \\ 1 & 1 & -1 & -2 & -4 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}$$

28. **[M]** The orthogonal basis is $f_0(t) = 1$, $f_1(t) = \cos t$, $f_2(t) = \cos^2 t - \frac{1}{2}$, and $f_3(t) = \cos^3 t - \frac{3}{4}\cos t$. Note that $2f_2(t) = \cos 2t$ and $4f_3(t) = \cos 3t$.

Section 6.8, page 437

2. Let X be the original design matrix, and let $\mathbf{y}$ be the original observation vector. Let W be the weighting matrix for the first method. Then the weighting matrix for the second method is $2W$. The weighted least-squares by the first method is equivalent to the ordinary least-squares for an equation whose normal equation is

$$(WX)^T WX\hat{\boldsymbol{\beta}} = (WX)^T W\mathbf{y} \qquad (1)$$

while the second method is equivalent to the ordinary least-squares for an equation whose normal equation is

$$(2WX)^T(2W)X\hat{\boldsymbol{\beta}} = (2WX)^T(2W)\mathbf{y} \qquad (2)$$

Since equation (2) can be written as $4(WX)^T WX\hat{\boldsymbol{\beta}} = 4(WX)^T W\mathbf{y}$, it has the same solutions as equation (1).

4. a. The vectors of polynomial values are

$$p_0 \leftrightarrow (1, 1, 1, 1, 1, 1), \quad p_1 \leftrightarrow (-5, -3, -1, 1, 3, 5),$$

$$p_2 \leftrightarrow (5, -1, -4, -4, -1, 5)$$

Verify that these vectors in $\mathbb{R}^6$ are mutually orthogonal.

b. $4p_0 + \frac{5}{7}p_1 + \frac{1}{14}p_2$

6. Use the identity

$$\sin mt \cos nt = \tfrac{1}{2}[\sin(mt + nt) + \sin(mt - nt)]$$

8. $-1 + \pi - 2\sin t - \sin 2t - \frac{2}{3}\sin 3t$

10. $\dfrac{4}{\pi}\sin t + \dfrac{4}{3\pi}\sin 3t$

12. The trigonometric identity $\cos 3t = 4\cos^3 t - 3\cos t$ shows that

$$\cos^3 t = \tfrac{3}{4}\cos t + \tfrac{1}{4}\cos 3t$$

The expression on the right is in the subspace spanned by the trigonometric polynomials of order 3 or less, so this expression *is* the third-order Fourier approximation to $\cos^3 t$.

14. g and h are both in the subspace H spanned by the trigonometric polynomials of order 2 or less. Since h is the second-order Fourier approximation to f, it is closer to f than any other function in the subspace H.

16. **[M]** $f_4(t) = \dfrac{4}{\pi}\sin t + \dfrac{4}{3\pi}\sin 3t$, $f_5(t) = f_4(t) + \dfrac{4}{5\pi}\sin 5t$

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16. **[M]** $\dfrac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = .00212$, $\text{cond}(A) \times \dfrac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 3363 \times (.00212) \approx 7.1$. In this case, $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ is almost the same as $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$, even though the large condition number suggests that $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ could be much larger.

18. **[M]** $\text{cond}(A) \times \dfrac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 23683 \times (1.097 \times 10^{-5}) = .2598$. This calculation shows that the relative change in $\mathbf{x}$, for this particular $\mathbf{b}$ and $\Delta\mathbf{b}$, should not exceed .2598. As it turns out, $\|\Delta\mathbf{x}\|/\|\mathbf{x}\| = .2597$. So the theoretical maximum change is almost achieved.

CHAPTER 7

Section 7.1, page 448

2. Not symmetric **4.** Symmetric **6.** Not symmetric

8. Orthogonal, $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

10. Not orthogonal

12. Orthogonal, $\begin{bmatrix} .5 & -.5 & .5 & -.5 \\ .5 & .5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$

14. $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$

16. $P = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}, D = \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix}$

18. $P = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$

20. $P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}, D = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

22. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix},$

$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix},$

$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

25. a. True. See Theorem 2 and the paragraph preceding the theorem.

b. True. This is a particular case of the statement in Theorem 1, when $\mathbf{u}$ and $\mathbf{v}$ are nonzero.

c. False. There are n real eigenvalues (Theorem 3), but they need not be distinct (Example 3).

d. False. See the paragraph following formula (2), in which each $\mathbf{u}$ is a unit vector.

26. a. True, by Theorem 2.

b. True. See the displayed equation in the paragraph before Theorem 2.

c. False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. The matrix P in Example 2 is an orthogonal matrix, but it is not symmetric.

d. True, by Theorem 3(b).

28. $(A\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x}^TA\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$, because $A^T = A$.

30. If A and B are orthogonally diagonalizable, then A and B are symmetric, by Theorem 2. If $AB = BA$, then $(AB)^T = (BA)^T = A^TB^T = AB$. So AB is symmetric and hence is orthogonally diagonalizable, by Theorem 2.

32. If $A = PRP^{-1}$, then $P^{-1}AP = R$. Since P is orthogonal, $R = P^TAP$. Hence $R^T = (P^TAP)^T = P^TA^TP^{TT} = P^TAP = R$, which shows that R is symmetric. Since R is also upper triangular, its entries above the diagonal must be zeros, to match the zeros below the diagonal. Thus R is a diagonal matrix.

34. $A = 7\mathbf{u}_1\mathbf{u}_1^T + 7\mathbf{u}_2\mathbf{u}_2^T - 2\mathbf{u}_3\mathbf{u}_3^T$, where

$$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix},$$

$$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} 1/18 & -4/18 & -1/18 \\ -4/18 & 16/18 & 4/18 \\ -1/18 & 4/18 & 1/18 \end{bmatrix}, \text{ and}$$

$$\mathbf{u}_3\mathbf{u}_3^T = \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix}.$$

36. Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Suppose $B^T = B$ and $B^2 = B$. Then $B^TB = BB = B$.

a.
$$\begin{aligned}\mathbf{z}\cdot\hat{\mathbf{y}} &= (\mathbf{y} - B\mathbf{y})\cdot(B\mathbf{y}) = \mathbf{y}\cdot(B\mathbf{y}) - (B\mathbf{y})\cdot(B\mathbf{y}) \\ &= \mathbf{y}^TB\mathbf{y} - (B\mathbf{y})^TB\mathbf{y} = \mathbf{y}^TB\mathbf{y} - \mathbf{y}^TB^TB\mathbf{y} = 0\end{aligned}$$

So $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

b. Any vector in $W = \operatorname{Col} B$ has the form $B\mathbf{u}$ for some $\mathbf{u}$. To show that $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $B\mathbf{u}$, we can use Exercise 28 since B is symmetric:

$$(\mathbf{y} - \hat{\mathbf{y}})\cdot B\mathbf{u} = [B(\mathbf{y} - \hat{\mathbf{y}})]\cdot\mathbf{u} = [B\mathbf{y} - BB\mathbf{y}]\cdot\mathbf{u} = 0$$

because $B^2 = B$. So $\mathbf{y} - \hat{\mathbf{y}}$ is in $W^\perp$, and the decomposition $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$ expresses $\mathbf{y}$ as the sum of a vector in W and a vector in $W^\perp$. By the Orthogonal Decomposition Theorem in Section 6.3, this decomposition is unique, and so $\hat{\mathbf{y}}$ must be $\operatorname{proj}_W \mathbf{y}$.

37. [M] $P = \dfrac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$,

$$D = \begin{bmatrix} 18 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

38. [M] $P = \begin{bmatrix} .8 & -.2 & .4 & -.4 \\ .4 & -.4 & -.2 & .8 \\ .4 & .4 & -.8 & -.2 \\ .2 & .8 & .4 & .4 \end{bmatrix}$,

$$D = \begin{bmatrix} .25 & 0 & 0 & 0 \\ 0 & .30 & 0 & 0 \\ 0 & 0 & .55 & 0 \\ 0 & 0 & 0 & .75 \end{bmatrix}$$

39. [M] $P = \begin{bmatrix} -.4472 & 0 & .8 & -.4 \\ 0 & -.4472 & .4 & .8 \\ .8944 & 0 & .4 & -.2 \\ 0 & .8944 & .2 & .4 \end{bmatrix}$,

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

Note: .4472 is an approximation to $1/\sqrt{5}$.

40. [M] $P =$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & -3/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & -4/\sqrt{20} & 1/\sqrt{5} \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & -28 & 0 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$$

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2. a. $4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$
b. 21 **c.** 5

4. a. $\begin{bmatrix} 20 & 7.5 \\ 7.5 & -10 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & .5 \\ .5 & 0 \end{bmatrix}$

6. a. $\begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

8. $P = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, $\mathbf{y}^T D\mathbf{y} = 15y_1^2 + 9y_2^2 + 3y_3^2$

In Exercises 10–14, other answers (change of variables and new quadratic form) are possible.

10. Positive definite; eigenvalues are 11 and 1

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

New quadratic form: $11y_1^2 + y_2^2$

12. Negative definite; eigenvalues are -1 and -6

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

New quadratic form: $-y_1^2 - 6y_2^2$

14. Indefinite; eigenvalues are 9 and -1

Change of variable: $\mathbf{x} = P\mathbf{y}$ where $P = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

New quadratic form: $9y_1^2 - y_2^2$

16. **[M]** Positive definite; eigenvalues are 7.5, 4.5, 3.5, .5

Change of variable: $\mathbf{x} = P\mathbf{y}$; $P = \begin{bmatrix} -.5 & -.5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \\ .5 & -.5 & .5 & -.5 \\ .5 & .5 & .5 & .5 \end{bmatrix}$

New quadratic form: $7.5y_1^2 + 4.5y_2^2 + 3.5y_3^2 + .5y_4^2$

18. **[M]** Indefinite; eigenvalues are 17, 1, -1, -7

Change of variable: $\mathbf{x} = P\mathbf{y}$;

$$P = \begin{bmatrix} -3/\sqrt{12} & 0 & 0 & 1/2 \\ 1/\sqrt{12} & 0 & -2/\sqrt{6} & 1/2 \\ 1/\sqrt{12} & -1/\sqrt{2} & 1/\sqrt{6} & 1/2 \\ 1/\sqrt{12} & 1/\sqrt{2} & 1/\sqrt{6} & 1/2 \end{bmatrix}$$

New quadratic form: $17y_1^2 + y_2^2 - y_3^2 - 7y_4^2$

20. 5

21. **a.** True, by the definition before Example 1, even though a nonsymmetric matrix could be used to compute values of a quadratic form.

b. True. See the paragraph following Example 3.

c. True, because the columns of P in Theorem 4 are eigenvectors of A. Review the Diagonalization Theorem (Theorem 5) in Section 5.3.

d. False. $Q(\mathbf{x}) = 0$ when $\mathbf{x} = \mathbf{0}$.

e. True. Theorem 5(a).

f. True. See the Numerical Note after Example 6.

22. **a.** True. See the paragraph before Example 1.

b. False. The matrix P must be orthogonal and make P^TAP diagonal. See the paragraph before Example 4.

c. False. There are also "degenerate" cases: a single point, two intersecting lines, or no points at all. See the subsection "A Geometric View of Principal Axes."

d. False. See the definition before Theorem 5.

e. True, by Theorem 5(b). If $\mathbf{x}^TA\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^TA\mathbf{x}$ is negative definite.

24. If $\det A > 0$, then by Exercise 23, $\lambda_1\lambda_2 > 0$, so that λ_1 and λ_2 have the same sign; also, $ad = \det A + b^2 > 0$.

a. If $\det A > 0$ and $a > 0$, then $d > 0$, too (because $ad > 0$). By Exercise 23, $\lambda_1 + \lambda_2 = a + d > 0$. Since λ_1 and λ_2 have the same sign, they are both positive. So Q is positive definite, by Theorem 5.

b. If $\det A > 0$ and $a < 0$, then $d < 0$, too. As in (a), we conclude that λ_1 and λ_2 are both negative and that Q is negative definite.

c. If $\det A < 0$, then by Exercise 23, $\lambda_1\lambda_2 < 0$, which shows that λ_1 and λ_2 have opposite signs. By Theorem 5, Q is indefinite.

26. We may assume $A = PDP^T$, with $P^T = P^{-1}$. The eigenvalues of A are all positive; denote them by $\lambda_1, \dots, \lambda_n$. Let C be the diagonal matrix with $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ on the diagonal. Then $D = C^2 = C^TC$. If $B = PCP^T$, then B is positive definite because its eigenvalues are the positive numbers on the diagonal of C. Also,

$$\begin{aligned} B^TB &= (PCP^T)^T(PCP^T) = (P^{TT}C^TP^T)(PCP) \\ &= PC^TCP \qquad \text{Because } P^TP = I \\ &= PDP = A \end{aligned}$$

28. The eigenvalues of A are all positive, by Theorem 5. Since the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A (see Exercise 25 in Section 5.1), the eigenvalues of A^{-1} are positive. (Note that A^{-1} is symmetric.) By Theorem 5, the quadratic form $\mathbf{x}^TA^{-1}\mathbf{x}$ is positive definite.

Section 7.3, page 465

2. $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$

4. **a.** 5 **b.** $\pm\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ **c.** 2

6. a. $\frac{15}{2}$ **b.** $\pm\begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$ **c.** $\frac{5}{2}$

8. Any unit vector that is a linear combination of $\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix}$ and $\begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$. Equivalently, any unit vector that is orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

10. $1+\sqrt{17}$

12. Let $\mathbf{x}$ be a unit eigenvector for the eigenvalue λ. Then $\mathbf{x}^TA\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda$, because $\mathbf{x}^T\mathbf{x} = 1$. So λ must satisfy $m \le \lambda \le M$.

14. [M] a. 17 **b.** $\begin{bmatrix}.5\\.5\\.5\\.5\end{bmatrix}$ **c.** 13

16. [M] a. 9 **b.** $\begin{bmatrix}-2/\sqrt{6}\\0\\1/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$ **c.** 3

Section 7.4, page 475

2. 5, 0 **4.** 3, 1

6. $\begin{bmatrix}-2&0\\0&-1\end{bmatrix} = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

8. $\begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}\\1/\sqrt{5}&2/\sqrt{5}\end{bmatrix}\begin{bmatrix}4&0\\0&1\end{bmatrix}\begin{bmatrix}1/\sqrt{5}&2/\sqrt{5}\\-2/\sqrt{5}&1/\sqrt{5}\end{bmatrix}$

10. $\begin{bmatrix}2/\sqrt{5}&1/\sqrt{5}&0\\1/\sqrt{5}&-2/\sqrt{5}&0\\0&0&1\end{bmatrix}\begin{bmatrix}5&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}\\1/\sqrt{5}&2/\sqrt{5}\end{bmatrix}$

12. $\begin{bmatrix}1/\sqrt{3}&1/\sqrt{2}&1/\sqrt{6}\\1/\sqrt{3}&0&-2/\sqrt{6}\\1/\sqrt{3}&-1/\sqrt{2}&1/\sqrt{6}\end{bmatrix}\begin{bmatrix}\sqrt{3}&0\\0&\sqrt{2}\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$

14. From Exercise 7, $A = U\Sigma V^T$ with $V = \begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}\\1/\sqrt{5}&2/\sqrt{5}\end{bmatrix}$. The first column of V is a unit vector at which $\|A\mathbf{x}\|$ is maximized.

16. a. rank $A = 2$

b. Basis for Col A: $\left\{\begin{bmatrix}-.86\\.31\\.41\end{bmatrix}, \begin{bmatrix}-.11\\.68\\-.73\end{bmatrix}\right\}$

Basis for Nul A: $\begin{bmatrix}.65\\.08\\-.16\\-.73\end{bmatrix}, \begin{bmatrix}-.34\\.42\\-.84\\-.08\end{bmatrix}$

18. The determinant of an orthogonal matrix U is ±1, because

$$1 = \det I = \det U^TU = (\det U^T)(\det U) = (\det U)^2$$

Suppose A is square and $A = U\Sigma V^T$. Then Σ is square, and

$$\begin{aligned}\det A &= (\det U)(\det\Sigma)(\det V^T)\\ &= \pm\det\Sigma = \pm\sigma_1\cdots\sigma_n\end{aligned}$$

20. The singular values of A are the square roots of the eigenvalues of A^TA. But $A^TA = A^2$ because A is symmetric. The eigenvalues of A^2 are $\lambda_1^2, \dots, \lambda_n^2$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Since the λ_i are positive, the square root of λ_i^2 is λ_i itself.

22. The right singular vector $\mathbf{v}_1$ is an eigenvector for the largest eigenvalue λ_1 of A^TA. By Theorem 7 in Section 7.3, the second largest eigenvalue, λ_2, is the maximum of $\mathbf{x}^T(A^TA)\mathbf{x}$ over all unit vectors orthogonal to $\mathbf{v}_1$. Since $\mathbf{x}^T(A^TA)\mathbf{x} = \|A\mathbf{x}\|^2$, the square root of λ_2, which is the second singular value of A, is the maximum of $\|A\mathbf{x}\|$ over all unit vectors orthogonal to $\mathbf{v}_1$.

24. From Exercise 23, $A^T = \sigma_1\mathbf{v}_1\mathbf{u}_1^T + \cdots + \sigma_r\mathbf{v}_r^T\mathbf{u}_r^T$. Then

$$\begin{aligned}A^T\mathbf{u}_j &= (\sigma_j\mathbf{v}_j\mathbf{u}_j^T)\mathbf{u}_j && \text{Because } \mathbf{u}_i^T\mathbf{u}_j = 0 \text{ for } i \ne j\\ &= \sigma_j\mathbf{v}_j && \text{Because } \mathbf{u}_j^T\mathbf{u}_j = 1\end{aligned}$$

26. [M] $\begin{bmatrix}.5&-.5&-.5&-.5\\.5&.5&.5&-.5\\.5&-.5&.5&.5\\.5&.5&-.5&.5\end{bmatrix}\begin{bmatrix}40&0&0&0\\0&20&0&0\\0&0&10&0\\0&0&0&0\end{bmatrix}$
$\times\begin{bmatrix}-.4&.8&-.2&.4\\.8&.4&.4&.2\\.4&-.2&-.8&.4\\-.2&-.4&.4&.8\end{bmatrix}$

The entries in this exercise are simple, to allow students to check their work mentally or by hand. The *Study Guide* contains a sequence of MATLAB commands that produce this SVD.

28. [M] 27.3857, 12.0914, 2.61163, .00115635; $\sigma_1/\sigma_4 = 23{,}683$

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2. $M = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$; $B = \begin{bmatrix} -3 & 1 & -2 & 2 & 3 & -1 \\ -6 & 2 & -3 & -1 & 6 & 2 \end{bmatrix}$,

$S = \begin{bmatrix} 5.6 & 8 \\ 8 & 18 \end{bmatrix}$

4. $\begin{bmatrix} .44 \\ .90 \end{bmatrix}$ for $\lambda = 21.9$, $\begin{bmatrix} -.90 \\ .44 \end{bmatrix}$ for $\lambda = 1.7$

6. **[M]** $y_1 = .62x_1 + .60x_2 + .51x_3$, which explains 64.9% of the total variance.

8. $y_1 = .44x_1 + .90x_2$; y_1 explains 92.9% of the variance.

10. **[M]** $c_1 = .41$, $c_2 = .82$, $c_3 = .41$ to two decimal places, or $c_1 = 1/\sqrt{6}$, $c_2 = 2/\sqrt{6}$, $c_3 = 1/\sqrt{6}$. The variance of y is 15.

12. By Exercise 11, the change of variable $\mathbf{X} = P\mathbf{Y}$ changes the covariance matrix S of $\mathbf{X}$ into the covariance matrix P^TSP of $\mathbf{Y}$. The total variance of the data, as described by $\mathbf{Y}$ is $\text{tr}(P^TSP)$. However, since P^TSP is similar to S, they have the same trace (Exercise 25 in Section 5.4). Thus the total variance of the data is unchanged by this change of variable.

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16. **[M]** $A^+ = \begin{bmatrix} .5 & 0 & -.05 & -.15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & .50 & 1.50 \\ .5 & -1 & -.35 & -1.05 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\hat{\mathbf{x}} = \begin{bmatrix} 2.3 \\ 0 \\ 5.0 \\ -.9 \\ 0 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Adding any nonzero vector $\mathbf{u}$ in Nul A to $\hat{\mathbf{x}}$ changes a zero entry to a nonzero entry; in this case the inequality $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$ is evident.